Wavefunction Mereology

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Abstract

I propose and defend an analysis of the parthood relation among quantum wavefunctions. Basically, according to the analysis, one wavefunction is part of another just in case the latter can be projected onto the former. Strikingly, the analysis implies many standard mereological principles. And the analysis also implies interesting, compelling answers to familiar questions about parthood. So in all these ways and more, this paper illustrates the benefits of bringing contemporary mereology into closer contact with contemporary physical theorizing.

1 Introduction

There are three big themes in this paper. The first and most straightforward theme: wavefunctions have parts. In what follows, I formulate an analysis of the parthood relation among wavefunctions. Roughly put, according to that analysis, one wavefunction is part of another just in case there is a special sort of map—a certain projection—from the latter to the former; the map is defined in terms of standard notions from quantum theories. This analysis, which I call 'Projective Part', implies that parthood satisfies many standard principles of classical mereology.

The second theme of this paper: the study of metaphysics in general, and the study of mereology in particular, stands to benefit enormously from engaging with more contemporary theories of actual-world physics. This is not to say that metaphysicians, or mereologists, deserve criticism for their degree of engagement so far. Rather, this is just to say that a certain kind of engagement with contemporary physics can pay serious metaphysical dividends. For instance, as this paper demonstrates, such engagement reveals overlooked arguments for old mereological views. Such engagement also demonstrates how standard principles of mereology—like reflexivity, anti-symmetry, transitivity, and various supplementation principles—can actually be derived from analyses of parthood based on quantum theories. And such engagement helps bring mereological theorizing into more contact with actual-world physics, which—unlike many examples of composition, which focus on classical particles—is distinctively non-classical.

The third theme of this paper: there is room for metaphysics in general, and mereology in particular, in philosophical discussions of contemporary physics. Projective Part provides an analysis of parthood which requires only wavefunctions and Hilbert spaces. That analysis, in turn, can be used to locate tables, chairs, and other medium-sized dry goods, in the wavefunction. This is not to say that other proposed methods for locating those items—such as those based on decoherence (Wallace, 2012), or those based on Humean features of quantum dynamics (Albert, 2015)—are somehow misguided. Rather, this is just to say that contemporary metaphysics can support worthwhile, but so far overlooked, approaches to locating those items in wavefunctions.

Overall, this paper embraces the spirit of (Wallace, 2020). For I propose an approach to parthood that is more sensitive to realistic physics than approaches largely motivated by classical theories of particles. In addition, Projective Part is perfectly compatible with the wavefunctions described by contemporary quantum field theories; though, to keep things accessible, I do not discuss the details of those theories here. And the ideas underlying Projective Part can be extended to other kinds of physical quantum states beyond just wavefunctions: for instance, those ideas can be extended to realist views of densities matrices (Wallace & Timpson, 2010); though again, for the sake of brevity, I do not discuss that here.

Before diving in, it is worth providing some additional motivation for the project of exploring wavefunction mereology. That additional motivation is simple: both philosophers and physicists often claim that wavefunctions have parts of some sort, and it is worth exploring what the basis for those claims could be. For instance, Wallace writes that wavefunctions have various functional structures as parts (2003, p. 102; 2012, p. 71; p. 74; p. 197). Physicists routinely describe some quantum states as parts of others (French & Taylor, 1978, p. 148; Griffiths, 2014, p. 142; Lancaster & Blundell, 2014, p. 100; Peskin & Schroeder, 1995, p. 634; Shankar, 1994, p. 15). Similarly, as Wallace reports, physicists often take measurement devices to be parts of quantum states (Wallace, 2012, p. 20; Wallace, 2020, p. 70). Schaffer suggests that wavefunctions have parts, when arguing for priority monism (2010, p. 52). Maudlin points out problems that arise for standard mereology—formulated for objects in three-dimensional space—given the holistic characters of wavefunctions (1998, p. 55). And Calosi, Fano, and Tarozzi provide many arguments for the view that wavefunctions have parts (Calosi et al., 2011; Calosi & Tarozzi, 2014).

All this discussion of wavefunction parts raises a question: what would a good theory of the parthood relation, among wavefunctions, look like? This paper provides an answer. Projective Part is a good theory of what the parts of wavefunctions are.

Relatedly, the wavefunctions discussed in this paper are ontological. Perhaps wavefunctions are fundamental, or perhaps not; regardless, in this paper, I adopt the view that wavefunctions are physical items of some sort or other. I do so because in standard mereology, the parthood relation obtains between bits of ontology. So if wavefunctions are laws, or somehow mental or epistemic, then it is unclear whether wavefunctions can stand in the parthood relation. But if wavefunctions are ontological, then it makes more sense to think that wavefunctions have parts.[1]

Here is a quick overview of the theory to come. In Section 2, I introduce the basic mathematical tools which feature heavily in this paper, and then I formulate Projective Part.

[1]For discussion of all these different views of what wavefunctions are, see (Chen, 2019; Miller, 2014; Myrvold, 2015).

I also discuss some of Projective Part's extremely attractive features: as mentioned earlier, for instance, Projective Part implies many principles of standard mereology. In Section 3, I use Projective Part to formulate some arguments for a few different views in the mereology literature. Specifically, Projective Part can be used to formulate arguments for views about atoms and something similar to gunk. Finally, in Section 4, I show that Projective Part implies an interesting kind of mereological universalism.

Two final remarks. First, there is simply not enough space to thoroughly explain all aspects of the formalism upon which Projective Part is based. So I take for granted that the reader is roughly familiar with at least some such formalism: the mathematics of probability amplitudes, the intuitive relationship between wavefunctions and the Born rule and the outcomes of experiments, and so on.[2]

Second, this paper contains several formal results: for instance, derivations of those standard mereological principles. I summarize those results in the body of the paper. The derivations are in an appendix.

2 Parthood

In this section, I formulate Projective Part and then discuss some of its implications for mereology. To start, I introduce the basic notions which Projective Part invokes. Then I state Projective Part. Finally, I discuss some of its most attractive featuers, and I defend it against several objections.

By way of preparation, I introduce some standard mathematical terms from quantum theories. To start, a Hilbert space is a vector space over which an operation of vector multiplication—called an 'inner product'—has been defined.[3] The inner product satisfies a series of formal conditions that, intuitively, one would expect multiplication to satisfy:

[2]For excellent introductions to all this, see (Albert, 1994; Shankar, 1994; Wallace, 2012).

[3]Moreover, that operation can be used to define a norm, relative to which the Hilbert space in question is complete.

basically, for instance, it distributes over sums.

The range of an inner product is the set of elements to which the inner product function maps pairs of vectors in a Hilbert space. That set can be any field, associated with the Hilbert space in question, at all. For our purposes, however, the field is the set of complex numbers. For that is the field used most often in quantum theories.

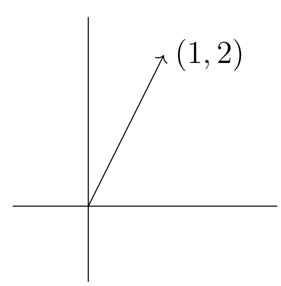
Here is some notation. Let $\mathcal{H}$ be a Hilbert space. Symbols of the form $|x\rangle$, $|y\rangle$, and so on, are vectors in $\mathcal{H}$. And for any vectors $|x\rangle$ and $|y\rangle$ in $\mathcal{H}$, the inner product of $|y\rangle$ with $|x\rangle$ is written $\langle y|x\rangle$. Since this inner product is a complex number, it follows that—like all such numbers—vectors can be multiplied by it. The vector $|z\rangle$ in $\mathcal{H}$, for instance, can be multiplied by the complex number $\langle y|x\rangle$; this is written $\langle y|x\rangle\,|z\rangle$.

Before explaining, in an intuitive way, what inner products represent, some more terms must be introduced. A 'basis', for a Hilbert space $\mathcal{H}$, is a subset of vectors $\mathcal{B}$ in $\mathcal{H}$—called 'basis vectors'—with a very special property. Roughly put, the property is this. Take any vector $|x\rangle$ in $\mathcal{H}$ whatsoever. Then there are vectors $|b_1\rangle, |b_2\rangle, \ldots$, in $\mathcal{B}$, such that for some complex numbers $c_1, c_2, \ldots$, the following holds.

$$|x\rangle = c_1\,|b_1\rangle + c_2\,|b_2\rangle + \cdots$$

In other words, every vector in $\mathcal{H}$ can be written as a sum—perhaps finite, or perhaps infinite—of constant multiples of basis vectors. Each such vector can be decomposed, that is, into vectors in the basis $\mathcal{B}$.

For example, take the standard $x - y$ plane which gets discussed in middle school and high school.



This plane is a Hilbert space. Every point in the plane is a vector; the point $(1,2)$ is the vector represented in the picture by an arrow from the origin. The vectors $(1,0)$ and $(0,1)$—that is, the unit vectors along the x-axis and y-axis respectively—form a basis for the whole space. In other words, every vector in this space can be expressed as a constant times $(1,0)$ plus a constant times $(0,1)$. The vector $(1,2)$, for instance, can be written like this: $(1,2) = 1 \cdot (1,0) + 2 \cdot (0,1)$.

In quantum theories, the vast majority of Hilbert spaces have bases which are countable and orthonormal.[4] Intuitively, $\mathcal{B}$ is a countable, orthonormal basis for Hilbert space $\mathcal{H}$ just in case $\mathcal{B}$ is countable and the following two conditions hold for all basis vectors $|b_i\rangle$ and $|b_j\rangle$ in $\mathcal{B}$.

(i) If $i \neq j$, then $|b_i\rangle$ and $|b_j\rangle$ are perpendicular.

(ii) The length of $|b_i\rangle$ is one.

Rigorous versions of both conditions can be formulated using inner products.[5]

In what follows, adopt a fixed enumeration of the vectors in $\mathcal{B}$, so that $\mathcal{B} = \{|b_1\rangle, |b_2\rangle, \ldots\} = \{|b_i\rangle\}_{i\in\mathbb{N}}$. With this notational convention, the very special property of $\mathcal{B}$ can be expressed like this. Take any vector $|x\rangle$ in $\mathcal{H}$. Then as a simple but tedious proof shows, there exists a unique smallest set $A \subseteq \mathbb{N}$ such that for some set $\{c_i\}_{i\in A}$ of complex numbers,

$$|x\rangle = \sum_{i\in A} c_i |b_i\rangle \tag{1}$$

Because $A \subseteq \mathbb{N}$ is the 'smallest' set for which an expression of the form in equation (1) can be written, it can be proved that each c_i is non-zero. This is important for proofs, given in the appendix, of various theorems that Projective Part implies.

There is a natural and intuitive description of how vectors, certain corresponding basis vectors, and certain corresponding inner products, all relate to one another. Take any vector

[4] That is, most Hilbert spaces in quantum theories are separable. For a recent example of a non-separable Hilbert space in a quantum theory, see (Bachelot, 2022).

[5] The rigorous version of (i) is: for basis vectors $|b_i\rangle \neq |b_j\rangle$, $\langle b_i|b_j\rangle = 0$. And the rigorous version of (ii) is: for basis vector $|b_i\rangle$, $\sqrt{\langle b_i|b_i\rangle} = 1$.

$|x\rangle$ in a Hilbert space $\mathcal{H}$ with a countable and orthonormal basis $\mathcal{B}$, and take any basis vector $|b_i\rangle$. Then the complex number $\langle b_i|x\rangle$ represents 'how much' of $|x\rangle$ extends along $|b_i\rangle$. And the vector $\langle b_i|x\rangle\,|b_i\rangle$ represents the chunk of $|x\rangle$ which extends in the $|b_i\rangle$ direction. Put a little more formally, $\langle b_i|x\rangle\,|b_i\rangle$ is the projection, onto $|b_i\rangle$, of $|x\rangle$. To see all this, it helps to consider the picture below.

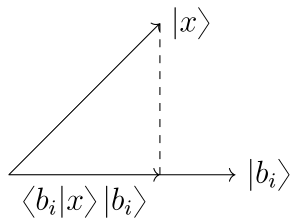


Figure 1: projection onto a basis vector.

The diagonal arrow represents $|x\rangle$. The longer arrow pointing straight to the right represents $|b_i\rangle$. The shorter arrow pointing straight to the right represents $\langle b_i|x\rangle\,|b_i\rangle$. So as the picture illustrates, the inner product of $|x\rangle$ with $|b_i\rangle$—that is, $\langle b_i|x\rangle$—represents 'how much' of $|x\rangle$ extends in the direction of $|b_i\rangle$. The vector representing that extension is $\langle b_i|x\rangle\,|b_i\rangle$; this vector is, intuitively, the chunk of $|x\rangle$ which extends in that direction. And as the dotted line shows, $\langle b_i|x\rangle\,|b_i\rangle$ is, intuitively, the result of projecting $|x\rangle$ onto $|b_i\rangle$.

As a simple proof shows, these chunks of $|x\rangle$—in the directions of various basis vectors—can be used to rewrite equation (1) in a more useful form. In particular, take any vector $|x\rangle$ in $\mathcal{H}$. Then there exists a unique smallest set $A \subseteq \mathbb{N}$ such that

$$|x\rangle = \sum_{i\in A} \langle b_i|x\rangle\,|b_i\rangle \tag{2}$$

In other words, $|x\rangle$ is just the sum of its projections along basis vectors.

Before formulating Projective Part, one more definition is needed. For each $S \subseteq \mathbb{N}$, let P_S be the function which takes any vector $|x\rangle$ and projects that vector onto the set of basis vectors index by S. A little more formally, for each $S \subseteq \mathbb{N}$, let P_S be the function defined as follows: for each vector $|x\rangle$,

$$P_S |x\rangle = \sum_{i \in S} \langle b_i | x \rangle |b_i\rangle$$

Each P_S is called a projection operator. For an example of one, just look at Figure 1. The function which takes any given $|x\rangle$ and maps it to $\langle b_i|x\rangle |b_i\rangle$ is a projection operator. That operator represents projection onto the $|b_i\rangle$ vector.

Think of these projections as maps into subspaces. For each $S \subseteq \mathbb{N}$, the set of basis vectors $\{|b_i\rangle\}_{i \in S}$ generates a very special set of vectors in $\mathcal{H}$: in particular, by taking lots of sums and products, the vectors in $\{|b_i\rangle\}_{i \in S}$ generate a Hilbert space that lies within $\mathcal{H}$. So P_S projects each vector $|x\rangle$ into the subspace formed by $\{|b_i\rangle\}_{i \in S}$.

With all that as background, here is Projective Part. Let $\mathcal{H}$ be the Hilbert space of some quantum theory. So the vectors in $\mathcal{H}$ represent physical wavefunctions. Let $\mathcal{B}$, which is used to define the projections, be a specific basis—countable and orthonormal—for $\mathcal{H}$. For present purposes, it is especially sensible to let $\mathcal{B}$ be a 'position space' basis: so intuitively, the vectors in $\mathcal{B}$ represent position configurations of particles. Then Projective Part is as follows.

Projective Part

For all non-null $|x\rangle$ and $|y\rangle$ in $\mathcal{H}$,[6] $|x\rangle$ is *part* of $|y\rangle$ if and only if there exists an $S \subseteq \mathbb{N}$ such that $P_S |y\rangle = |x\rangle$.[7]

In other words, the parts of a wavefunction. are all and only the projections of that wavefunction into various subspaces.

For example, take a non-null wavefunction $|y\rangle$ whose unique decomposition is $|y\rangle = \sum_{i \in A} \langle b_i|y\rangle |b_i\rangle$ for some smallest $A \subseteq \mathbb{N}$. Suppose that there are two distinct natural numbers

[6] A non-null vector is just a non-zero vector. Every Hilbert space contains exactly one vector whose inner product with every other vector is zero. That is the null vector; all other vectors are non-null.

[7] For brevity, this formulation of Projective Part ignores complications relating to normalizations of the wavefunctions in question. Those facts about normalization can be incorporated into what Projective Part says; the resulting principle is just cumbersome to state.

m and n in A, and consider the wavefunction $|x\rangle = \langle b_n|y\rangle\,|b_n\rangle + \langle b_m|y\rangle\,|b_m\rangle$.[8] Then as a simple proof shows, there is an $S \subseteq \mathbb{N}$ such that $P_S\,|y\rangle = |x\rangle$; just take $S = \{m, n\}$. So Projective Part implies that wavefunction $|x\rangle$ is part of wavefunction $|y\rangle$.

A quick disclaimer: in this paper, whenever I use expressions like "the parts of the wavefunction" or "this vector is part of that vector," I do not mean that one mathematical object is part of another. Those expressions should be read as shorthand for "the parts of the physical object which the wavefunction represents" or "this vector represents this physical state, that vector represents that physical state, and the former physical state is part of the latter physical state." For the mathematical wavefunctions and mathematical vectors used to represent physical states—in quantum theories—are not the sorts of things that have parts (Wallace, 2012, p. 287). So my claims about the parts of wavefunctions, or vectors, are just convenient shorthand for claims about the parts of the physical items which those wavefunctions and vectors represent.

Note that Projective Part provides an analysis of parthood given a particular choice of basis $\mathcal{B}$. As is well-known, Hilbert spaces like $\mathcal{H}$ contain many different countable, orthonormal bases. Just as there is a basis $\mathcal{B}$ whose vectors represent the positions of particles – for short, position states – there is also a basis $\mathcal{B}'$ whose vectors represent the momenta of particles – for short, momentum states. Consequently, just as Projective Part analyzes parthood in terms of projections defined using $\mathcal{B}$, another principle could analyze parthood in terms of projections defined using $\mathcal{B}'$. And likewise for the many, many other bases, whose vectors represent states of many other sorts.

This raises several interesting questions. Does this show that parthood for wavefunctions is pluralistic, in that there is one kind of parthood for each basis of the relevant Hilbert space? If so, then should parthood for wavefunctions be understood as a three-place relation that obtains between (i) a basis, (ii) a wavefunction, and (iii) another wavefunction? If not, then which kind of parthood—for which basis—is the basic, real, or most fundamental, one?

[8] As a trivial proof shows, $|x\rangle$ is non-null.

For a variety of reasons, I think that parthood for wavefunctions is not pluralistic. And for a variety of reasons, I think that the basic—or real, or most fundamental—kind of parthood is the one captured by Projective Part; that is, the one analyzed in terms of the position basis $\mathcal{B}$. The reason is simple: $\mathcal{B}$ is the basis whose vectors correspond most clearly to the positions of individual particles, and so $\mathcal{B}$ is most apt to capture mereological relations. Besides, many different interpretations of quantum mechanics endorse some sort of preferred basis, in order to solve various problems; and often, the preferred basis is $\mathcal{B}$ (Albert, 1992; North, 2021; Wallace, 2012).[9] So there is nothing unusual in using $\mathcal{B}$ as a preferred basis for wavefunction mereology.

Now for some standard mereological terminology. Take any non-null $|x\rangle$ and $|y\rangle$ in $\mathcal{H}$. Say that $|x\rangle$ is a 'proper part' of $|y\rangle$ just in case $|x\rangle$ is part of $|y\rangle$ and $|x\rangle \neq |y\rangle$. And say that $|x\rangle$ 'overlaps' $|y\rangle$ just in case there is a non-null $|z\rangle$ in $\mathcal{H}$ such that $|z\rangle$ is part of $|x\rangle$ and $|z\rangle$ is part of $|y\rangle$.

For four different reasons, Projective Part has lots of intuitive appeal; and these are significant reasons to like Projective Part. First, given a standard view of how to define ordinary objects using wavefunctions, Projective Part implies all the usual parthood relations among ordinary objects. An example, based on a table, will illustrate why. According to the standard view, the table just is a subvector that the wavefunction of the world contains.[10] In particular, the table is the vector $|t\rangle$ given by summing projections of the wavefunction $|x\rangle$ along basis vectors corresponding to the degrees of freedom of the particles that, intuitively, comprise the table. Now, because table is a vector in $\mathcal{H}$, Projective Part implies that the

[9]There is lots of debate over why a given basis counts as preferred, of course (Barrett, 2005; Hemmo & Shenker, 2022). But in order to defend these answers to the interesting questions above, I need not adopt any particular account of where the preference, for a given basis, comes from. Perhaps the specialness of $\mathcal{B}$ is explained by decoherence, or perhaps that specialness is just a primitive fact about the physical structure of the universe, or perhaps something else explains that specialness. Regardless, all these answers require is that $\mathcal{B}$ is a uniquely natural basis with which to analyze parthood. And that is independent of exactly why $\mathcal{B}$ is uniquely natural.

[10]This standard view is perhaps most plausible on Everettian interpretations of quantum mechanics and GRW interpretations of quantum mechanics. Many versions of the Bohmian interpretation would identify the table not with a subvector of the wavefunction, but rather, with a subset of the particles that obey the guidance equation.

table has parts. Moreover, the parts of the table are themselves vectors that correspond to collections of the table's constituent particles. According to the standard view, that is, the table's back left leg is just a vector $|b\rangle$ given by summing projections of the table $|t\rangle$ along basis vectors corresponding to the degrees of freedom of the particles that, intuitively, comprise the leg in question. And according to Projective Part, $|b\rangle$ – that is, the table's back left leg – is part of $|t\rangle$ – that is, the table.

This is an extremely attractive feature of Projective Part. Basically, given a standard view of how to define ordinary objects using wavefunctions, Projective Part implies all the usual parthood relations among those objects. And that is a serious point in favor of Projective Part.

Second, Projective Part implies that each vector $\langle b_i|x\rangle\,|b_i\rangle$, in equation (2), is part of $|x\rangle$. And intuitively, that is the right result. Each complex number $\langle b_i|x\rangle$ represents 'how much' of $|x\rangle$ extends in the $|b_i\rangle$ direction. Each vector $\langle b_i|x\rangle\,|b_i\rangle$ represents the chunk of $|x\rangle$ that extends along $|b_i\rangle$. And in addition, each $\langle x|b_i\rangle\,|b_i\rangle$ features in the unique decomposition of $|x\rangle$ given by (2). So in a natural and intuitive sense, each of these $\langle b_i|x\rangle\,|b_i\rangle$ are the basic chunks from which $|x\rangle$ is built. In other words, intuitively, $|x\rangle$ contains each $\langle b_i|x\rangle\,|b_i\rangle$ as a part. And Projective Part captures that.

Third, and relatedly, Projective Part basically says that the parts of a wavefunction $|x\rangle$ are all and only the sums of wavefunctions which take the form $\langle b_i|x\rangle\,|b_i\rangle$. As was just explained, each $\langle b_i|x\rangle\,|b_i\rangle$ is part of $|x\rangle$: for each $\langle b_i|x\rangle\,|b_i\rangle$ is the chunk of $|x\rangle$ that extends along $|b_i\rangle$, and moreover, $|x\rangle$ can be decomposed uniquely as a sum of these $\langle b_i|x\rangle\,|b_i\rangle$. It makes sense to say that sums of those chunks—those parts of $|x\rangle$—are themselves parts of $|x\rangle$ too. And that is precisely what Projective Part says.

Fourth, projections of wavefunctions into subspaces generated by position basis wavefunctions are, intuitively, maps from collections of particles to collections of subsets of those particles. This results, in large part, from the fact that $\mathcal{B}$ is the position basis. So projections of wavefunctions are maps from (i) amplitudes of configurations for all of the particles in the universe, to (ii) the corresponding amplitudes of configurations for just some of the particles in the universe. So projections are basically maps which 'delete' some wavefunction-related facts about some particles, while preserving all the other wavefunction-related facts about all the other particles. Put another way, projections are basically formal tools for taking the universe as a whole and examining a small, particulate subset of it. And intuitively, the particles in that subset comprise part of the particles of the universe as a whole. And that, basically, is what Projective Part captures.

In addition to its intuitive appeal, Projective Part is attractive for other, quite striking reasons. Basically, as a series of proofs show, Projective Part implies that the parthood relation is reflexive, anti-symmetric, transitive, and conforms to three standard supplementation principles. In what follows, I summarize these results; for the proofs, see the appendix.

To start, here are the first three principles which Projective Part implies. They say that parthood is reflexive, anti-symmetric, and transitive.

<u>Reflexivity</u>

Let $|x\rangle$ be a non-null vector in $\mathcal{H}$. Then $|x\rangle$ is part of $|x\rangle$.

<u>Anti-Symmetry</u>

Let $|x\rangle$ and $|y\rangle$ be non-null vectors in $\mathcal{H}$. Suppose that $|x\rangle$ is part of $|y\rangle$ and $|y\rangle$ is part of $|x\rangle$. Then $|x\rangle = |y\rangle$.

<u>Transitivity</u>

Let $|x\rangle$, $|y\rangle$, and $|z\rangle$ be non-null vectors in $\mathcal{H}$. Suppose that $|x\rangle$ is part of $|y\rangle$ and $|y\rangle$ is part of $|z\rangle$. Then $|x\rangle$ is part of $|z\rangle$.

Altogether, Reflexivity, Anti-Symmetry, and Transitivity imply that given Projective Part, the parthood relation is a non-strict partial order over the wavefunctions represented by vectors in $\mathcal{H}$.

In addition, Projective Part implies the following supplementation principle. Roughly put, this principle says that if one thing is not part of another thing, then there is something whose parts are exactly those parts of the former thing which do not overlap the latter thing.

Complementation

Let $|x\rangle$ and $|y\rangle$ be non-null vectors in $\mathcal{H}$. Suppose that $|x\rangle$ is not part of $|y\rangle$. Then there exists a non-null vector $|z\rangle$ in $\mathcal{H}$ such that for all non-null vectors $|w\rangle$ in $\mathcal{H}$, $|w\rangle$ is part of $|z\rangle$ if and only if

(i) $|w\rangle$ is part of $|x\rangle$, and

(ii) $|w\rangle$ and $|y\rangle$ do not overlap.

Principles like Complementation imply the Strong Supplementation principle[11] and the Weak Supplementation principle[12] of classical mereology. So Projective Part, by implying Complementation, implies those other principles too.

All this is an extremely attractive feature of Projective Part. Basically, a large chunk of classical mereology follows from Projective Part directly. And that is a significant point in its favor.

Because of this, Projective Part supports the claims about parthood made by Calosi, Fano, and Tarozzi (2011). As those authors discuss, it makes sense to claim that the parthood relation among quantum states is reflexive, anti-symmetric, and transitive. And it makes sense to claim that the parthood relation among quantum states conforms to various supplementation principles. Those authors do not derive those claims from a specific, proposed analysis of parthood; they argue, instead, that these are the sorts of claims to which a parthood relation among quantum states should conform. The mereological theory given by

[11] Roughly put, Strong Supplementation says that if one thing is not part of another, then there is something which is part of the former and which does not overlap the latter.

[12] Roughly put, Weak Supplementation says that if one thing is a proper part of another, then the latter has a part which does not overlap the former.

Projective Part is compatible with those arguments; and moreover, Projective Part actually implies the claims that Calosi, Fano, and Tarozzi defend.

One might object to Projective Part as follows. Take the basis vector $|b_1\rangle$, and consider the two vectors $2\,|b_1\rangle$ and $4\,|b_1\rangle$. Projective Part implies, perhaps surprisingly, that these two vectors do not overlap. The reason: the same basis vector b_1 is associated with two different coefficients in those vectors—with 2 in $2\,|b_1\rangle$, and with 4 in $4\,|b_1\rangle$—and given Projective Part, it follows that $2\,|b_1\rangle$, and $4\,|b_1\rangle$ do not have any parts in common.[13] But one might object that this is bizarre. The vector $4\,|b_1\rangle$ is just two times the vector $2\,|b_1\rangle$, so surely these vectors overlap. In other words, because these vectors are constant multiples of one another, they must share a part in common: perhaps they both contain $2\,|b_1\rangle$ as a part, for instance. So Projective Part is false.

There are two problems with this objection. First, and most simply, it overlooks the fact that at most one of the vectors $4\,|b_1\rangle$ and $2\,|b_1\rangle$ is part of the actual world. To see why, let the actual world's wavefunction be $|a\rangle$. Then Projective Part implies that $|a\rangle$ contains at most one of $4\,|b_1\rangle$ and $2\,|b_1\rangle$ as a part. So it makes no sense to reject Projective Part for the reason that this objection does—namely, that according to Projective Part, $2\,|b_1\rangle$ is not part of $4\,|b_1\rangle$—because only one of $4\,|b_1\rangle$ and $2\,|b_1\rangle$ is actual. Rejecting Projective Part for the reason that this objection does is like rejecting some other analysis of parthood which fails to imply that horns in the actual world are not parts of unicorns: since there are no unicorns, analyses of parthood should not imply that unicorns contain actual-world horns as parts. That makes no sense. But that is entirely analogous to rejecting Projective Part simply because it implies that $2\,|b_1\rangle$ is not part of $4\,|b_1\rangle$: at most one of those two wavefunctions is part of the actual world, so of course it is not the case that one should contain the other as a part.

Second, this objection is based on a misunderstanding of the notion of parthood at

[13] To put it slightly more precisely, as proved in the appendix, Projective Part implies that in order for any two vectors $|x\rangle$ and $|y\rangle$ to overlap, there must be a basis vector $|b_i\rangle$ such that for some complex number c, $c\,|b_i\rangle$ is part of $|x\rangle$ and $c\,|b_i\rangle$ is part of $|y\rangle$; and if $|x\rangle = 2\,|b_1\rangle$ and $|y\rangle = 4\,|b_2\rangle$, then there is no such basis vector and no such complex number.

issue here. It is true, of course, that in some intuitive sense of 'overlap', constant non-zero multiples of the same basis vector overlap one another. But this notion of overlap, and the corresponding notion of parthood, is not mereological; at least, not in the standard sense. For as it turns out, there is no good way to analyze parthood such that (i) vectors like $2\,|b_1\rangle$ count as parts of vectors like $4\,|b_1\rangle$, and yet (ii) the analysis respects the standard principles of mereology.[14] Making matters worse, all of the candidate analyses lack the intuitive appeal that Projective Part has. Let us see why.

To start, consider this analysis: non-null $|x\rangle$ is part of non-null $|y\rangle$ if and only if $|y\rangle$ is a constant non-zero multiple of $|x\rangle$. This analysis implies that $2\,|b_1\rangle$ is part of $4\,|b_1\rangle$. But this analysis violates Anti-Symmetry and Complementation; in fact, it violates both Strong Supplementation and Weak Supplementation as well. In addition, this analysis lacks the intuitive appeal of Projective Part, because this analysis does not imply that the wavefunction for any given subsystem of particles is part of the wavefunction for all the particles in the universe; in fact this analysis implies that the former sorts of wavefunctions are practically never parts of the latter sorts of wavefunctions. So this analysis is a non-starter.

Similarly, take another analysis: non-null $|x\rangle$ is part of non-null $|y\rangle$ if and only if $\langle x|y\rangle \neq 0$. This analysis implies that $2\,|b_1\rangle$ is part of $4\,|b_1\rangle$. But this analysis violates Anti-Symmetry, Transitivity and Complementation; in fact, it violates both Strong Supplementation and Weak Supplementation as well. In addition, this analysis implies that any non-orthogonal vectors are parts of one another; and that amounts to countenancing way too many instances of the parthood relation. So this analysis, too, is a non-starter.[15]

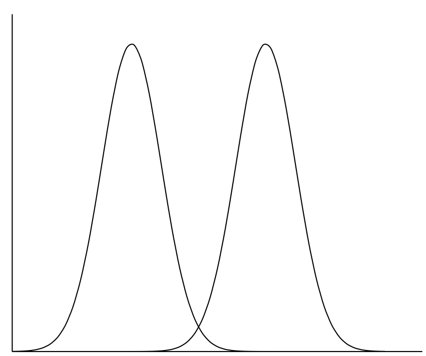
For all these reasons, the objection fails. Perhaps it is true that in some sense of 'overlap'—based on an algebraic notion of parthood, say—constant non-zero multiples of the

[14] In what follows, I omit the proofs of the claims that I make about these alternative analyses: for those proofs are trivial, but also extremely tedious.

[15] For analogous reasons, similar analyses are also problematic. For instance, consider an analysis which says that vectors are parts of those vectors to which they sum. That is, consider an analysis like this: non-null $|x\rangle$ is part of non-null $|y\rangle$ if and only if for some $|z\rangle$, $|y\rangle = |z\rangle + |x\rangle$. This analysis implies that every non-null vector is part of every non-null vector: everything is part of everything, in other words. So this analysis completely trivializes the notion of parthood.

same basis vector overlap one another. But that is not the mereological sense of overlap relevant here. So that notion of overlap cannot support an objection to Projective Part.

Alternatively, one might object to Projective Part by making some observations about wavefunction amplitudes. Basically, the objection is this. Suppose two wavefunctions assign extremely close amplitudes to one and the same basis vector: so the amplitudes assigned, despite being numerically distinct, are very nearly the same. To illustrate, consider the two wavefunctions in the picture below.



Suppose that the wavefunctions assign extremely close—yet still slightly different—amplitudes to the states corresponding to the middle region, where the wavefunctions' bell curves intersect. Then Projective Part implies the following: in that region, the wavefunctions do not overlap mereologically. In other words, despite assigning very nearly the same amplitudes to the basis vectors corresponding to states in that region, the wavefunctions do not share a common part there. And that seems bizarre. Because the amplitudes of the wavefunctions in that region are so close, one might claim that those amplitudes should be considered more-or-less the same; they certainly are the same for all practical purposes, anyway. So intuitively, the wavefunctions share a part there. And so Projective Part is problematic, one might claim, since it contradicts that.

There are two good responses to this objection. The first response simply rejects the key intuition upon which the objection is based: it is not the case that wavefunctions should count as overlapping mereologically when they assign numerically distinct amplitudes to the same basis vector, regardless of how close those amplitudes may be. For vectors of the form $c\,|b_i\rangle$—for c a complex number and $|b_i\rangle$ a basis vector—can be thought of as ascribing a property to an object. The object—or to put it a bit more perspicuously, the degree of freedom—corresponds to $|b_i\rangle$. The property corresponds to the probability amplitude represented by c. Now suppose that $|b_i\rangle$ is a basis vector representing one of the states within the intersecting region in the picture above. Let c and c' be extremely close, yet numerically distinct, complex numbers. Suppose that the wavefunction on the left contains $c\,|b_i\rangle$ as a part, while the wavefunction on the right contains $c'\,|b_i\rangle$ as a part. Then these wavefunctions ascribe different properties to the object corresponding to $|b_i\rangle$: the property represented by c, and the property represented by c'. So despite the fact that c and c' are very close, we should reject the objection's claim that intuitively, these wavefunctions share a part here. For the wavefunctions there assign different properties to the same object.

The second response is more conciliatory. Basically, it turns out that Projective Part can be revised so as to satisfy the key intuition reported in the objection. The revision, which introduces thresholds, is roughly this: so long as the amplitudes which wavefunctions assign to basis vectors representing states in a region are 'close enough'—in the sense that the differences between the amplitudes are less than some given threshold—then the revised version of Projective Part implies that these wavefunctions overlap in that region.[16] For lack

[16]The revision, which invokes the notion of a norm on a vector space $||\cdot||$—defined in terms of the inner product—is as follows.

> ϵ-Threshold Projective Part
> Let $\epsilon > 0$. For all non-null $|x\rangle$ and $|y\rangle$ in $\mathcal{H}$, $|x\rangle$ is *part* of $|y\rangle$ if and only if there exists an $S \subseteq \mathbb{N}$ such that $||\,|x\rangle - P_S\,|y\rangle\,|| < \epsilon$.

In other words, one vector is part of another just in case the distance—as measured by the norm—between the first and some projection of the second is less than a given ϵ. The main problem with this analysis concerns the ϵ which it invokes: it is unclear what makes it the case that a particular value of ϵ yields the correct parthood relation among vectors. For suppose that ϵ is determined by context, so that ϵ can vary from one context to the next. Then by ϵ-Threshold Projective Part, parthood is determined by context too. But that is strange: facts about parthood are usually taken to be a context-independent. Alternatively, suppose that there is some worldly, objective value for ϵ. Then that is strange too: it is strange to think that the world somehow determines a specific value of ϵ as the correct threshold for parthood, rather than a value which is marginally greater or less than that.

of space, I will not discuss this analysis in detail here. But Projective Part is, basically, a simplified version of this more complicated analysis.

Before continuing, it is worth briefly mentioning a nice bonus of Projective Part: it fills a gap in certain metaphysical arguments concerning the ontological priority relation between parts and wholes. In particular, while arguing for priority monism—the view that the whole is ontologically prior to its parts—Schaffer invokes parts of wavefunctions (2010). Schaffer does not provide an account of what wavefunction parts are, however. But Projective Part does. And Projective Part is compatible with the other premises of Schaffer's argument. So Projective Part fills a gap in a standard argument for priority monism in the literature.

3 Atoms and Quasi-Gunk

In this section, I use Projective Parthood to argue for specific mereological views connected to atoms and gunk. According to one view, everything—or at least, every wavefunction—has an atom as a part. According to another view, some things are what I call 'quasi-gunky'; not quite gunky, but similar to gunk in certain respects.

These arguments illustrate what mereologists stand to gain from studying contemporary physical theories. By exploring parthood in the quantum realm, mereologists can obtain new insight into well-worn disputes. For the analysis of parthood supported by quantum theories—namely, Projective Part—can help us adjudicate contentious mereological views in the literature.

Before continuing, some terminology. Take any non-null $|x\rangle$ in $\mathcal{H}$. Say that $|x\rangle$ is an 'atom' just in case for all $|z\rangle$ in $\mathcal{H}$, $|z\rangle$ is not a proper part of $|x\rangle$.[17] For each non-null $|y\rangle$

[17]Since all contemporary quantum theories are non-fundamental, the wavefunctions which those theories describe are probably non-fundamental too. So ultimately, these wavefunction atoms may well have parts, in the sense that these atoms may be generated by more fundamental physical items. In what follows, however, I make the simplifying assumption that wavefunction atoms can indeed be atomic in the mereological sense. For that assumption may well be vindicated by some future physical theory. And regardless, it is worth developing the theory of how wavefunctions could have basic wavefunction parts—and it is convenient to call those basic parts 'atoms'—even if those basic parts ultimately turn out to have parts themselves.

in $\mathcal{H}$, say that $|x\rangle$ is an 'atomic part' of $|y\rangle$ just in case $|x\rangle$ is an atom and $|x\rangle$ is part of $|y\rangle$. Say that $|x\rangle$ is 'gunky' just in case $|x\rangle$ has a proper part and every proper part of $|x\rangle$ itself has a proper part. And say that $|x\rangle$ is 'quasi-gunky' just in case there exist non-null $|y_1\rangle, |y_2\rangle, |y_3\rangle, \ldots$ in $\mathcal{H}$ such that $|y_1\rangle$ is a proper part of $|x\rangle$, $|y_2\rangle$ is a proper part of $|y_1\rangle$, $|y_3\rangle$ is a proper part of $|y_2\rangle$, and so on.

Now for the first argument. Projective Part implies the following principle concerning mereological atoms.

Atomicity

Let $|x\rangle$ be a non-null vector in $\mathcal{H}$. Then there is a non-null vector $|y\rangle$ in $\mathcal{H}$ such that $|y\rangle$ is an atom and $|y\rangle$ is part of $|x\rangle$.

So Projective Part settles a long-standing debate in the mereological literature. For Projective Parthood implies the view that everything contains atoms as parts; that is simply what Atomicity says.

Next, the second argument. Projective Part implies the following principle concerning quasi-gunk.

Quasi-Gunk

There is a non-null, quasi-gunky vector $|x\rangle$ in $\mathcal{H}$.

In other words, there are infinite descending chains of proper parthood: $|x\rangle$ has a proper part $|y_1\rangle$, which has a proper part $|y_2\rangle$, which has a proper part $|y_3\rangle$, and so on.[18] So Projective Part settles a long-standing debate in the mereological literature. For Projective Part implies the view that something is infinitely divisible into proper parts; that is simply what Quasi-Gunk says.

[18] As explained in the appendix, Quasi-Gunk only follows from Projective Part if the basis $\mathcal{B}$ is countably infinite. If $\mathcal{B}$ is finite, then there are no quasi-gunky vectors.

Of course, one might respond to these arguments by rejecting Projective Part. Perhaps Projective Part is not the correct analysis of what the parts of wavefunctions are. Perhaps wavefunction parts are not projections into subspaces.

The problem with this response, of course, is that it faces a difficult question. Suppose that Projective Part is not the correct analysis of what the parts of wavefunctions are. Then what is the correct analysis? What are the parts of wavefunctions? Those who wish to deny these arguments' conclusions must supply answers to these questions. So those who reject these conclusions have their work cut out for them. They need to propose an alternative analysis of parthood for wavefunctions, and then show that their alternative analysis does not imply anything like Atomicity or Quasi-Gunk. And that looks hard to do: just consider how some alternative analyses of parthood, from Section 2, were utterly implausible.

Lewis asked whether reality consists entirely of atoms, or entirely of gunk, or some of each; and then lamented that his own theory did not provide an answer to this question (1991, p. 74). Projective Parthood provides an answer, at least for the case where the things in question are wavefunctions. Every wavefunction consists entirely of atoms—that is, every wavefunction contains atoms as parts—but in addition, some wavefunctions are quasi-gunky, in the sense that they contain proper parts, which contain proper parts, and so on.

So Projective Part helps us make progress on some long-standing debates in mereology. There has been lots of discussion over whether something like Atomicity holds, or something like Quasi-Gunk holds, or both, or neither (Hudson, 2007; Markosian, 1998; Russell, 2008; Sider, 2013; Zimmerman, 1996). For the most part, those debates do not engage with the ontologies suggested by quantum theories. Projective Part, however, does. And in so doing, Projective Part has important implications for that debate: there are atoms and there is quasi-gunk.

It is worth noting that for at least two reasons, Projective Part does not settle these debates once and for all. First, the fundamental physical theory of the world is not yet known. As far as anyone can tell, that theory will probably be formulated in terms of wavefunctions evolving in Hilbert spaces. But that could turn out to be wrong. And if so, then of course, Projective Part is not the correct theory of parthood for fundamental physics.

Second, sometimes these mereological debates concern objects that exist, but are neither wavefunctions nor wavefunction parts. Many debates concern the mereological structure of spacetime, for instance. Of course, it is not yet clear whether spacetime—or at least, the kind of spacetime which appears in these debates—actually exists; that is an empirical question, to be decided by the fundamental physical theory of the world. But if so, then there is more in the world than just the wavefunction and its parts: there is spacetime too. And it could turn out that the mereological structure of spacetime is not atomistic, or not quasi-gunky, in the way that the mereological structure of wavefunctions is.

Nevertheless, Projective Part still helps illuminate these mereological debates. It shows that one of the most basic, elementary objects posited by quantum theories—namely, the wavefunction—has atomic parts, and also can be quasi-gunky. Regardless of whether there is more to parthood than that, Projective Part tells us a great deal about the mereological structure of our world.

4 Universalism

Projective Part implies a composition principle akin to mereological universalism. In this section, I present that principle. Basically, it says that all collections of actual-world wavefunction parts compose something.

To start, here is the definition of the composition relation. Take any non-null $|x\rangle, |y_1\rangle, |y_2\rangle, \ldots$ in $\mathcal{H}$. Then say that the $|y_1\rangle, |y_2\rangle, \ldots$ 'compose' $|x\rangle$ just in case (i) each of $|y_1\rangle, |y_2\rangle, \ldots$ is part of $|x\rangle$, and (ii) every part of $|x\rangle$ overlaps one of the $|y_1\rangle, |y_2\rangle, \ldots$. In other words, some vectors compose another just in case the latter is, intuitively, the smallest vector which contains the former as parts.

Now for the composition principle which Projective Part implies. It says that some vectors compose another just in case, very roughly put, none of the former contain atoms whose basis vectors are the same but whose associated complex numbers are different.

Universal

Let $|x\rangle, |y_1\rangle, |y_2\rangle, \ldots$ be non-null vectors in $\mathcal{H}$. Then the $|y_1\rangle, |y_2\rangle, \ldots$ compose $|x\rangle$ if and only if

(i) every atomic part of a vector among the $|y_1\rangle, |y_2\rangle, \ldots$ is part of $|x\rangle$,

(ii) every atomic part of $|x\rangle$ is an atomic part of a vector among the $|y_1\rangle, |y_2\rangle, \ldots$, and

(iii) for each complex number c and each basis vector $|b_i\rangle$ such that $c\,|b_i\rangle$ is part of $|x\rangle$, and for each $|y_j\rangle$ among the $|y_1\rangle, |y_2\rangle, \ldots$, either $\langle b_i|y_j\rangle = c$ or $\langle b_i|y_j\rangle = 0$.

In other words, some vectors compose another vector just in case for every atom which is part of the latter, that atom is (i) part of at least one of the former vectors, and (ii) every other one of the former vectors either contains that atom as a part or is orthogonal to that atom.

Universal is quite intuitive. For orthogonal wavefunctions correspond to distinct subsystems of particles.[19] Pre-theoretically, those subsystems should be combinable; they should compose a larger subsystem of which they are both parts. And that is basically what Universal says, by way of the clauses about atoms.

Here is why Universal is a version of mereological universalism. One vector in $\mathcal{H}$ represents the wavefunction of the actual world; call it $|a\rangle$. Universal implies that for every non-null $|y_1\rangle, |y_2\rangle, \ldots$ in $\mathcal{H}$ which are parts of $|a\rangle$, there is a vector $|x\rangle$ in $\mathcal{H}$ such that (i) $|x\rangle$ is part of $|a\rangle$, and (ii) the $|y_1\rangle, |y_2\rangle, \ldots$ compose $|x\rangle$. In other words, Universal implies that every collection of actual-world parts composes something. And that is a version of

[19]Slightly more accurately: orthogonal wavefunctions correspond to distinct subsystems comprised of particles' degrees of freedom.

mereological universalism.

Note that Universal contradicts the view, called 'mereological nihilism', that only mereological atoms exist. For there exist orthogonal vectors in $\mathcal{H}$ which are parts of the actual wavefunction $|a\rangle$, and so which—according to Universal—compose something. This is, of course, a simple consequence of basic facts about Hilbert spaces.

As a brief aside, however, it is worth pointing out that a modified version of Projective Part implies a version of mereological nihilism.

<u>Nihilist Projective Part</u>

For all non-null $|x\rangle$ and $|y\rangle$ in $\mathcal{H}$, $|x\rangle$ is *part* of $|y\rangle$ if and only if there exists an $i \in \mathbb{N}$ and a complex number c such that $|y\rangle = |x\rangle = c\,|b_i\rangle$.

In other words, according to Nihilist Projective Part, one vector is part of another just in case (i) those vectors are identical, and (ii) those vectors are identical to a complex number multiplied by a basis vector. As a simple proof shows, Nihilist Projective Part implies that nothing non-trivially composes[20] anything else; so only atoms exist.

For a variety of reasons, versions of mereological universalism are preferable to versions of mereological nihilism; and so Projective Part is preferable to Nihilist Projective Part. Here is a particularly compelling reason. If a version of mereological nihilism holds for vectors, then there is no actual-world wavefunction. For the actual world, if it existed, would be an object corresponding to a sum of constant multiples of basis vectors; and given the version of mereological nihilism which Nihilist Project Part implies, that object does not exist. But of course, the actual world does exist. So mereological nihilism is false. And so mereological universalism is preferable to mereological nihilism.

In addition, for a variety of reasons, mereological universalism is generally preferable to more moderate views of composition too. I present a series of arguments for this in

[20]Nihilist Projective Part does allow composition to occur in the following, trivial cases: every constant multiple of a basis vector is part of itself.

(Wilhelm, 2020; 2022). But for now, roughly put, the idea of those arguments is this. There does not seem to be an exact, worldly boundary between (i) the collections of vectors that, intuitively, compose something, and (ii) the collections of vectors that, intuitively, do not compose anything. Any proposed boundary seems highly context-dependent and arbitrary. So the least arbitrary, and clearest, view of the matter is this: either composition always occurs, or composition never occurs. And so given the above arguments that mereological universalism is preferable to mereological nihilism, ultimately, mereological universalism is worth endorsing.

5 Conclusion

Projective Part is an extremely attractive analysis of parthood. It makes pre-theoretic sense, since it captures how the wavefunctions corresponding to subsystems of particles can be parts of the wavefunctions corresponding to supersystems of those particles. It implies that wavefunction parts satisfy many standard mereological principles: in particular, Reflexivity, Anti-Symmetry, Transitivity, and Complementation, and so both Strong Supplementation and Weak Supplementation. It can be used to illuminate debates over atoms and gunk. And it supports an attractively universal composition principle.

Moreover, as Projective Part demonstrates, there are noteworthy relationships between standard mereology and contemporary quantum theories. Those relationships have not received much attention to date; but they are worth studying. In fact, given Projective Part, much of standard mereology is actually a straightforward consequence of the physical posits which contemporary quantum theories make. That is striking. And it shows that a great deal stands to be gained, by taking contemporary quantum physics seriously when engaging in metaphysical debates.

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Appendix

In what follows, let $\mathcal{H}$ be a separable Hilbert space over field $\mathbb{F}$, and let $\mathcal{B}$ be a countable, orthonormal basis for $\mathcal{H}$. For convenience, adopt a fixed enumeration of the vectors in $\mathcal{B}$; so $\mathcal{B} = \{|b_i\rangle\}_{i\in\mathbb{N}}$. Also for convenience, all of the vectors mentioned in the statements of lemmas and theorems are non-null.

One more important point. Whenever I choose a set S so as to express a given $|x\rangle$ as a sum $|x\rangle = \sum_{i\in S} \langle b_i|x\rangle |b_i\rangle$, or perhaps as $|x\rangle = \sum_{i\in S} \langle b_i|y\rangle |b_i\rangle$ for some $|y\rangle$, the set S is always chosen to be 'smallest'. More precisely, S is always chosen so that there is no proper subset $S' \subset S$ such that $|x\rangle = \sum_{i\in S'} \langle b_i|x\rangle |b_i\rangle$, or $|x\rangle = \sum_{i\in S'} \langle b_i|y\rangle |b_i\rangle$. This ensures that S is chosen so that for all $i \in S$, $\langle b_i|x\rangle \neq 0$; and similarly, for all $i \in S$, $\langle b_i|y\rangle \neq 0$. And in what follows, say that this sum $\sum_{i\in S} \langle b_i|x\rangle |b_i\rangle$—where S is smallest—is the 'unique decomposition' of $|x\rangle$.

The following lemma says that each vector in $\mathcal{H}$ admits of a unique decomposition in terms of $\mathcal{B}$. Its proof, which can be found in any introductory linear algebra textbook, is omitted; for instance, see the result in and surrounding theorem 4.2 in (Lang 1986, p. 107).

Lemma 1. *Let $|x\rangle \in \mathcal{H}$, $A \subseteq \mathbb{N}$, and $A' \subseteq \mathbb{N}$. Let $\{c_i\}_{i\in A}$ and $\{d_{i'}\}_{i'\in A'}$ be sets of complex numbers, none of which are* 0. *Suppose that $|x\rangle = \sum_{i\in A} c_i |b_i\rangle$ and $|x\rangle = \sum_{i'\in A'} d_{i'} |b_{i'}\rangle$. Then $A = A'$ and for all $i \in A$, $c_i = d_i$*

Lemma 2. *Let $|y\rangle \in \mathcal{H}$ have unique decomposition $|y\rangle = \sum_{i\in A} \langle b_i|y\rangle |b_i\rangle$. Then for all $|x\rangle \in \mathcal{H}$, $|x\rangle$ is part of $|y\rangle$ if and only if there is an $S \subseteq A$ such that $|x\rangle = \sum_{i\in S} \langle b_i|y\rangle |b_i\rangle$.*

Proof. Because this lemma is—despite being simple—crucial for what follows, I include a detailed proof. For the left-to-right direction, suppose $|x\rangle$ is part of $|y\rangle$. Then Projective Part implies that there exists $S \subseteq \mathbb{N}$ such that $|x\rangle = P_S |y\rangle = \sum_{i\in S} \langle b_i|y\rangle |b_i\rangle$. Since $|y\rangle = \sum_{i\in A} \langle b_i|y\rangle |b_i\rangle$ it follows that for all $j \in S$ there is a $j' \in A$ such that $\langle b_j|y\rangle = \sum_{i\in A} \langle b_i|y\rangle \langle b_j|b_i\rangle = \langle b_{j'}|y\rangle$, where these equalities follow from the linearity of inner products and facts about bases. Therefore, by more elementary facts about bases—plus the fact that we are working with a fixed enumeration of $\mathcal{B}$—$j = j'$. Since this holds for all $j \in S$, it follows that $S \subseteq A$, which completes the proof of the left-to-right direction.

For the right-to-left direction, suppose that there is an $S \subseteq A$ such that $|x\rangle = \sum_{i\in S} \langle b_i|y\rangle |b_i\rangle$. Then since $S \subseteq \mathbb{N}$, and $P_S |y\rangle = \sum_{i\in S} \langle b_i|y\rangle |b_i\rangle = |x\rangle$, Projective Part implies that $|x\rangle$ is part of $|y\rangle$, which completes the proof of the right-to-left direction.

□

The next three theorems are standard postulates of classical mereology. They show that given Projective Part, parthood is reflexive, anti-symmetric, and transitive.

Theorem 1 (Reflexivity)**.** *Let $|x\rangle \in \mathcal{H}$ have unique decomposition $|x\rangle = \sum_{i\in A} \langle b_i|x\rangle |b_i\rangle$. Then $|x\rangle$ is part of $|x\rangle$.*

Proof. Follows immediately from lemma 2 applied to the case where $S = A$.

□

Theorem 2 (Anti-symmetry)**.** *Let $|x\rangle, |y\rangle \in \mathcal{H}$ have unique decompositions $|x\rangle = \sum_{i\in A} \langle b_i|x\rangle |b_i\rangle$ and $|y\rangle = \sum_{i\in A'} \langle b_i|y\rangle |b_i\rangle$. Suppose that $|x\rangle$ is part of $|y\rangle$ and $|y\rangle$ is part of $|x\rangle$. Then $|y\rangle = |x\rangle$.*

Proof. By lemma 2, there are $S' \subseteq A'$ and $S \subseteq A$ such that $|x\rangle = \sum_{i\in S'} \langle b_i|y\rangle |b_i\rangle$ and $|y\rangle = \sum_{i\in S} \langle b_i|x\rangle |b_i\rangle$. Lemma 1 implies both (a) $S' = A$ and for all $i \in S'$, $\langle b_i|x\rangle = \langle b_i|y\rangle$, and (b) $S = A'$ and for all $i \in S$, $\langle b_i|y\rangle = \langle b_i|x\rangle$. So $A = A'$. And from all this, it also follows that for all $i \in A$, $\langle b_i|x\rangle = \langle b_i|y\rangle$. So $|y\rangle = |x\rangle$.

□

Theorem 3 (Transitivity)**.** *Let $|x\rangle, |y\rangle, |z\rangle \in \mathcal{H}$ be such that the unique decompositions of $|y\rangle$ and $|z\rangle$ are $|y\rangle = \sum_{i\in A'} \langle b_i|y\rangle |b_i\rangle$ and $|z\rangle = \sum_{i\in A''} \langle b_i|z\rangle |b_i\rangle$. Suppose that $|x\rangle$ is part of $|y\rangle$ and $|y\rangle$ is part of $|z\rangle$. Then $|x\rangle$ is part of $|z\rangle$.*

Proof. Lemma 2 implies both (a) there exists $S' \subseteq A'$ such that $|x\rangle = \sum_{i\in S'} \langle b_i|y\rangle |b_i\rangle$, and (b) there exists $S'' \subseteq A''$ such that $|y\rangle = \sum_{i\in S''} \langle b_i|z\rangle |b_i\rangle$. By some algebra, $|x\rangle = \sum_{i\in S'\cap S''} \langle b_i|z\rangle |b_i\rangle$. Taking $S = S' \cap S''$ and applying lemma 2 yields that $|x\rangle$ is part of $|z\rangle$.

□

Lemma 3. *Let $|x\rangle, |y\rangle \in \mathcal{H}$ have unique decompositions $|x\rangle = \sum_{i\in A} \langle b_i|x\rangle |b_i\rangle$ and $|y\rangle = \sum_{i\in A'} \langle b_i|y\rangle |b_i\rangle$. Then $|x\rangle$ and $|y\rangle$ overlap if and only if there exists $j \in A \cap A'$ such that $\langle b_j|x\rangle = \langle b_j|y\rangle$.*

Proof. For the left-to-right direction, take non-null $|z\rangle \in \mathcal{H}$ such that $|z\rangle$ is part of both $|x\rangle$ and $|y\rangle$. Lemma 2 implies that there exists $S \subseteq A$ and $S' \subseteq A'$ such that $|z\rangle = \sum_{i\in S} \langle b_i|x\rangle |b_i\rangle$ and $|z\rangle = \sum_{i\in S'} \langle b_i|y\rangle |b_i\rangle$. Lemma 1 implies that $S = S'$ and for all $i \in S$, $\langle b_i|x\rangle = \langle b_i|y\rangle$. Take any $j \in S \subseteq A \cap A'$ to complete the proof of this direction.

For the right-to-left direction, take $j \in A \cap A'$ such that $\langle b_j|x\rangle = \langle b_j|y\rangle$. Define $|z\rangle = \langle b_j|x\rangle |b_j\rangle = \langle b_j|y\rangle |b_j\rangle$; note that $|z\rangle$ is non-null. Take $S = S' = \{j\}$ and apply lemma 2 to complete the proof.

□

The following theorem is a standard postulate of one widely endorsed version of classical mereology. It says that the complement of one vector, relative to another, always exists.

Theorem 4 (Complementation)**.** *Let $|x\rangle, |y\rangle \in \mathcal{H}$ have unique decompositions $|x\rangle = \sum_{i\in A} \langle b_i|x\rangle |b_i\rangle$ and $|y\rangle = \sum_{i\in A'} \langle b_i|y\rangle |b_i\rangle$. Suppose that $|x\rangle$ is not part of $|y\rangle$. Then there exists $|z\rangle \in \mathcal{H}$ such that for all $|w\rangle \in \mathcal{H}$, $|w\rangle$ is part of $|z\rangle$ if and only if*

(i) $|w\rangle$ is part of $|x\rangle$, and

(ii) $|w\rangle$ and $|y\rangle$ do not overlap.

Proof. Let $B = \{i \in A \mid \forall i' \in A', \langle b_i|x\rangle |b_i\rangle \neq \langle b_{i'}|y\rangle |b_{i'}\rangle\}$. To see that $B \neq \varnothing$, just note the following: if $B = \varnothing$ then for all $i \in A$ there is $i' \in A'$ such that $\langle b_i|x\rangle |b_i\rangle = \langle b_{i'}|y\rangle |b_{i'}\rangle$; but then letting $S' \subseteq A'$ be the set of all these i', writing $|x\rangle = \sum_{i \in A} \langle b_i|x\rangle |b_i\rangle = \sum_{i' \in S'} \langle b_{i'}|y\rangle |b_{i'}\rangle$, and applying lemma 2, implies that $|x\rangle$ is part of $|y\rangle$; contradiction. Define $|z\rangle = \sum_{i \in B} \langle b_i|x\rangle |b_i\rangle$. Note that $|z\rangle$ is non-null and part of $|x\rangle$.

Take non-null $|w\rangle \in \mathcal{H}$ with unique decomposition $|w\rangle = \sum_{i \in C} \langle b_i|w\rangle |b_i\rangle$. To start, suppose $|w\rangle$ is part of $|z\rangle$. Clearly, (i) follows from theorem 3. Since $|w\rangle$ is part of $|z\rangle$, lemma 2 implies that there exists $S \subseteq B$ such that $|w\rangle = \sum_{i \in S} \langle b_i|z\rangle |b_i\rangle$. Lemma 1 implies that $C = S$ and also this: for all $i \in C$, $\langle b_i|w\rangle = \langle b_i|z\rangle$, so $\langle b_i|w\rangle |b_i\rangle = \langle b_i|x\rangle |b_i\rangle$. Since $C = S \subseteq B$, it follows that for all $i \in C$ and all $i' \in A'$, $\langle b_i|w\rangle |b_i\rangle \neq \langle b_{i'}|y\rangle |b_{i'}\rangle$. So for all $i \in C \cap A'$, $\langle b_i|w\rangle \neq \langle b_i|y\rangle$. Along with lemma 3, (ii) follows.

Now suppose $|w\rangle$ is part of $|x\rangle$, and $|w\rangle$ and $|y\rangle$ do not overlap. By lemma 2, there exists $S \subseteq A$ such that $|w\rangle = \sum_{i \in S} \langle b_i|x\rangle |b_i\rangle$. Lemma 1 implies that $C = S$ and ultimately this: for all $i \in C$, $\langle b_i|w\rangle |b_i\rangle = \langle b_i|x\rangle |b_i\rangle$. As a simple but tedious paragraph of proof shows, $C \subseteq B$. So for all $i \in C$, $\langle b_i|w\rangle |b_i\rangle = \langle b_i|z\rangle |b_i\rangle$. Therefore, taking $S = C$ and applying lemma 2 establishes that $|w\rangle$ is part of $|z\rangle$.

□

Lemma 4 (Atoms)**.** *Let $|x\rangle \in \mathcal{H}$ have unique decomposition $|x\rangle = \sum_{i \in A} \langle b_i|x\rangle |b_i\rangle$. Then $|x\rangle$ is an atom if and only if for some non-zero $c \in \mathbb{F}$ and some $j \in \mathbb{N}$, $|x\rangle = c\,|b_j\rangle$.*

Proof. Follows from lemma 2.

□

Theorem 5. *For all $|x\rangle \in \mathcal{H}$ with unique decomposition $|x\rangle = \sum_{i \in A} \langle b_i|x\rangle |b_i\rangle$, and for all $j \in A$, $\langle b_j|x\rangle |b_j\rangle$ is part of $|x\rangle$.*

Proof. Taking $j \in A$ and letting $S = \{j\}$, and applying lemma 2, establishes this.

□

The following theorem is a standard postulate of one widely endorsed version of classical mereology. It says that everything contains an atom as a part.

Theorem 6 (Atomicity)**.** *For all $|x\rangle \in \mathcal{H}$ with unique decomposition $|x\rangle = \sum_{i \in A} \langle b_i|x\rangle |b_i\rangle$, there exists $|y\rangle \in \mathcal{H}$ such that $|y\rangle$ is an atom and $|y\rangle$ is part of $|x\rangle$.*

Proof. Follows from theorem 5.

□

Theorem 7 (Gunk)**.** *So long as $\mathcal{B}$ is infinite, there exists $|x\rangle \in \mathcal{H}$—with unique decomposition $|x\rangle = \sum_{i \in A} \langle b_i|x\rangle |b_i\rangle$—such that $|x\rangle$ is quasi-gunky.*

Proof. Choose $|x\rangle$ so that A is infinite, and let $\{B_j\}_{j \in \mathbb{N}}$ be such that $B_1 \subsetneq A$ and for all $j \in \mathbb{N}$, $B_{j+1} \subsetneq B_j$. Define $|y_j\rangle = \sum_{i \in B_j} \langle b_i|x\rangle |b_i\rangle$ for each $j \in \mathbb{N}$. Then lemma 2 implies that $|y_1\rangle$ is part of $|x\rangle$, and for all $j \in \mathbb{N}$, $|y_{j+1}\rangle$ is part of $|y_j\rangle$.

□

Theorem 8 (Universal)**.** *Let $|x\rangle \in \mathcal{H}$ have unique decomposition $|x\rangle = \sum_{i \in A} \langle b_i|x\rangle |b_i\rangle$. For either $\alpha \in \mathbb{N}$ or $\alpha = \omega$, let $J = \{1, \ldots, \alpha\}$. Let $\{|y_j\rangle\}_{j \in J} \subseteq \mathcal{H}$ be such that for all $j \in J$, the unique decomposition of $|y_j\rangle$ is $|y_j\rangle = \sum_{i \in A_j} \langle b_i|y_j\rangle |b_i\rangle$. Then the $\{|y_j\rangle\}_{j \in J}$ compose $|x\rangle$ if and only if all of the following obtain.*

(1) For all $j \in J$ and all $i_j \in A_j$ there exists $i \in A$ such that $\langle b_{i_j}|y_j\rangle |b_{i_j}\rangle = \langle b_i|x\rangle |b_i\rangle$.

(2) For all $i \in A$ there exists $j \in J$ such that $\langle b_i|x\rangle |b_i\rangle$ is part of $|y_j\rangle$.

(3) For all $i \in A$ and all $j \in J$, either $\langle b_i|y_j\rangle = \langle b_i|x\rangle$ or $\langle b_i|y_j\rangle = 0$.

Proof. Since the proof is a straightforward but tedious check that composition ultimately matches those three conditions—mostly following the basic idea of theorem 4—I omit it.

□

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