Meaning and Statespace

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Abstract

In this paper, I formulate a theory of how worldly states determine the truth values of sentences. Then I put that theory to work: I use it to analyze the contents of sentences, partial content, propositions, subject matters, entailment, counterfactuals, logical subtraction, and more. The theory draws from—and extends—recent work on state-based approaches to truth conditions and to various notions related to meaning, while avoiding some problems that other approaches face.

1 Introduction

What are sentences about? Given any particular sentence, what is its content? What does it take for the content of one sentence to be part of the content of another? How do the contents of simpler sentences determine the contents of sentences which are more complex? And what about other notions connected to meaning and semantics: propositions, subject matters, entailment, and so on? How should those be analyzed?

In this paper, I propose a theory—call it 'Statespace'—which supports answers to these questions.[1] Statespace is a particular implementation of a more general view of what makes

[1]There are different approaches to some—or most, or even all—of these questions, several of which I will discuss later. Some are based on syntactic features of denoting expressions (N. Goodman, 1961; Ryle, 1933). Others are based on sets, and partitions, of possible worlds (Lewis, 1986; 1998; Yablo, 2014). Still others appeal to structured complexes (King, 2007; Soames, 1987) or abstract algebras (Dorr, 2016; J. Goodman, 2019).

sentences true or false. According to that more general view, sentences are made true, or made false, by states of the world: the state of grass being green, for instance, makes the sentence "Grass is green" true.[2] Statespace implements this more general view, and in so doing, supports analyses of over a dozen semantic notions connected to the meanings of expressions in first-order logic.

In slogan form, Statespace says the following: states make sentences true, or false, in virtue of having parts which make the parts of those sentences true or false. To make true is to have parts which make certain sentential parts true; to make false is to have parts which make certain sentential parts false. That is the guiding idea behind Statespace.

In Section 2, I present the main conditions to which states conform: as will become clear, these conditions are mereological, in that they describe how some states are parts of other states. In Section 3, I use states to formulate truth conditions for sentences of first-order logic; along with the conditions in the previous section, these conditions comprise the theory Statespace.

Throughout the rest of the paper, I explore how Statespace can be used to analyze many different notions related to sentences and their contents. In Section 4, I use Statespace to analyze the contents of sentences, propositions, and three interrelated notions of partial content. In Section 5, I use Statespace to analyze the notion of making a sentence true in an exact way, the notion of making a sentence false in an exact way, the subject matters of sentences, and aboutness. In Section 6, I use Statespace to analyze relations of entailment and containment. In Section 7, I use Statespace to formulate truth conditions for counterfactuals. In Section 8, I use Statespace to analyze logical subtraction among propositions. Finally, in Section 9, I compare Statespace to other theories of states and truth in the literature.

The material to come gets quite technical. So to increase readability, in the main text, I present clear but often heuristic formulations of the key conditions, definitions, analyses,

[2] There are other particular implementations of this more general view—see (Barwise, 1981; Barwise & Perry, 1983; Fine, 2017a; 2017b; 2017c; Kratzer, 1989; 2012; van Fraassen, 1969)—and Statespace is similar to some of them. Throughout this paper, I discuss those similarities, while also discussing some crucial differences.

and theorems. For fully precise formulations of all that, see appendices A–G.

2 The Mereology of States

Some states are parts of others. In this section, I describe this parthood relation among states. To start, I discuss states in a little more detail. Then I present the mereological conditions to which states conform.

To see how states can be parts of each other, consider the following example. Take the state of grass being green, and take the state of roses being red. There is a state which combines these two. It is the state of grass being green and roses being red. And both of the previous two states are parts of this combination. The state of grass being green, in other words, is part of the state of grass being green and roses being red. The state of roses being red is part of the latter state too.

Following the literature, I take states to be fact-like entities (Elgin, 2021, p. 5; Fine & Jago, 2019, p. 536). For my purposes here, however, a state can be anything which obeys the principles to come. In particular, states can be whatever obeys the parthood conditions in this section and the semantic conditions in Section 3. Anything which does that is suited to play the semantic roles that states play in this paper.[3]

States have various modal features. For starters, only some states obtain. The state of grass being green obtains, for instance: it holds in the actual world. But the state of grass being white does not: it only obtains in non-actual possibilities. In addition, some states obtain necessarily. The state of four being even obtains in every possible world, for example, as does the state of five being prime. And some states necessarily fail to obtain. The state of four being odd—though it exists—does not obtain in any possible world whatsoever; and

[3] According to the view which I prefer—but which need not be combined with the coming postulates—the truth conditions presented in this paper jointly form a metaphysical semantics for sentences of first-order logic. The semantics is 'metaphysical', in the sense that it is the semantics by which rigorous metaphysical theories of the world should be interpreted; and it need not be the best semantics for linguistic theories of natural languages. For more on the notion of metaphysical semantics, see (Sider, 2011).

similarly for the state of grass being both green and white.

In a very rough and heuristic way, states may be described as parts of reality. Our world, for example, contains the state of grass being green as a part. Some merely possible worlds contain, as a part, the state of grass being white. But not all states are parts of some possible world or other. The state of four being odd is not part of any possible world, since it cannot obtain.[4]

Let us now consider the parthood conditions which describe relationships among states. Altogether, there are six of them. And as will become clear, they are pretty simple and intuitive. It is striking that such simple, intuitive conditions can support such a wide-ranging theory of the meanings of sentences in first-order logic.

These conditions invoke several technical symbols. For starters, they use a set of states: in what follows, represent that set by 'S'. In addition, they use a two-place predicate '$\sqsubseteq$' which represents the parthood relation: so for any states s and t in S, '$s \sqsubseteq t$' says that s is part of t.

Here are the first three conditions.

Reflexivity

For all s in S, $s \sqsubseteq s$.

Anti-Symmetry

For all s and t in S, if $s \sqsubseteq t$ and $t \sqsubseteq s$ then $s = t$.

Transitivity

For all r, s, and t in S, if $r \sqsubseteq s$ and $s \sqsubseteq t$ then $r \sqsubseteq t$.

[4]For lack of space, in this paper, I do not formulate theories of the interaction between states and modal notions like necessity and possibility; for some such theories, see (Angere, 2015; Elgin, 2021; Fine, 2017c; Moltmann, 2018).

Any relation on S, which satisfies these three conditions, is called a 'partial order'. So together, these three conditions say that $\sqsubseteq$ is a partial order over S.

The fourth condition says that there are no infinite descending chains of states.[5]

Well-foundedness

There do not exists states s_1, s_2, and so on, in S, such that $\ldots$, $s_3 \sqsubsetneq s_2$, $s_2 \sqsubsetneq s_1$.

In other words, there are no states such that (i) the first contains the second as a proper part,[6] (ii) the second contains the third as a proper part, (iii) the third contains the fourth as a proper part, and so on. Any relation on S, which satisfies this condition, is called 'well-founded'. So this condition, along with the previous three, says that $\sqsubseteq$ is a well-founded partial order over S.[7]

The last two conditions invoke the notions of least upper bound and greatest lower bound. Precise definitions of these notions are presented in Appendix A. For now, however, the following rough characterizations will suffice. For each set of states, an 'upper bound' of that set is a state which contains every state in that set as a part. And the 'least upper bound' of that set is, of all the upper bounds of that set, the smallest: it is smallest in the sense that it is, itself, part of every other upper bound. Similarly, for each set of states, a 'lower bound' of that set is a state which is part of every state in that set. And the 'greatest lower bound' of that set is, of all that set's lower bounds, the biggest: it is biggest in the sense that it contains, itself, every other lower bound as a part.

Now for the fifth condition. Basically, it says that given any collection of states, the

[5]In other words, states are not gunky. For discussions of gunk and its properties, see (Lewis, 1991; Russell, 2008; Zimmerman, 1996).

[6]One state is a 'proper part' of another just in case the former is part of the latter but the latter is not part of the former. Proper parthood is represented by the symbol '$\sqsubsetneq$'.

[7]Interestingly, this assumption is not actually needed. As a complicated proof shows, the truth conditions in Appendix B—along with some minimal additional assumptions about variable assignments and which objects exist—imply that the states which determine any given sentence's truth values do not form any infinite descending chains (the proof's complications stem from certain 'infinitary' characteristics of the truth conditions for first-order quantifiers). So I adopt this fourth condition merely as a simplifying assumption.

least upper bound of that collection exists.

Existence of Least Upper Bound

For each subset A of S, there exists a state in S which is the least upper bound of A; denote this state by '$\bigsqcap A$'.[8]

For instance, take the following two states: (i) the state of grass being green, and (ii) the state of roses being red. According to the condition above, there exists a smallest state which contains the states in both (i) and (ii). This smallest state is, of course, the state of grass being green and roses being red.

Least upper bounds are closely related to the more familiar notion of fusions. In particular, the least upper bound of some states is, intuitively, the fusion of those states. That is, the least upper bound of a set A of states is the fusion of the states in A. For the fusion of some entities is, in general, the smallest object which contains those entities as parts. And that is precisely what the least upper bound of the entities in A—namely, $\bigsqcap A$—is.

The sixth condition says that given any collection of states, the greatest lower bound of that collection exists.

Existence of Greatest Lower Bound

For each subset A of S, there exists a state in S which is the greatest lower bound of A; denote this state by '$\bigsqcup A$'.

[8]As will become clear, I use '$\sqcap$' and '$\bigsqcap$' as the symbols for least upper bounds, while I use '$\sqcup$' and '$\bigsqcup$' as the symbols for greatest lower bounds. This is the opposite of how those symbols are often used in the mathematics and philosophy literature: often, '$\sqcap$' and '$\bigsqcap$' are used for greatest lower bounds, while '$\sqcup$' and '$\bigsqcup$' are used for least upper bounds. I do not follow that convention because unfortunately, in the context of truth conditions, it is quite misleading. The operation of least upper bound is the state-theoretic analog of conjunction, while the operation of greatest lower bound is the state-theoretic analog of disjunction. So since conjunction is represented by the symbol '$\wedge$', it makes sense to represent least upper bounds using '$\sqcap$' and '$\bigsqcap$' rather than '$\sqcup$' and '$\bigsqcup$'. And since disjunction is represented by the symbol '$\vee$', it makes sense to represent greatest lower bounds using '$\sqcup$' and '$\bigsqcup$' rather than '$\sqcap$' and '$\bigsqcap$'.

For instance, take the following two states: (i) the state of grass being green, and (ii) the state of roses being red. According to the condition above, there exists a biggest state which is part of (i) and also part of (ii). This biggest state is, it turns out, the state of grass being green or roses being red.

In what follows, I adopt two notational conventions. First, for all r and s in S, the least upper bound of $\{r, s\}$ is denoted '$r \sqcap s$'. Second, for all r and s in S, the greatest lower bound of $\{r, s\}$ is denoted '$r \sqcup s$'.

Now for the final definition of this section. For any set S and any well-founded, partial order $\sqsubseteq$ over S, S and $\sqsubseteq$ jointly form a 'complete, well-founded lattice' just in case S and $\sqsubseteq$ jointly satisfy the last two conditions just mentioned. As will become clear, complete and well-founded lattices are the basis for the theory of meaning to come.

Altogether, this theory of states—call it the 'Complete Lattice' theory—is pretty simple. According to the Complete Lattice theory, parthood among states is reflexive, anti-symmetric, transitive, and well-founded. In addition, every collection of states has both a least upper bound and a greatest lower bound. And that is all. No more mereological assumptions will be needed, in order to give a comprehensive theory of meaning for sentences in first-order logic.[9]

3 Truth Conditions

In this section, I use states to formulate truth conditions. To start, I briefly summarize the first-order language on which I will focus. Then I present the truth conditions.

The language—call it 'L'—consists of the following symbols. First, L contains infinitely many constants 'a', 'b', 'c', and so on. Second, L contains infinitely many variables 'x', 'y', 'z', and so on. Third, for each natural number n, L contains infinitely many n-place predicates

[9] One caveat: in order to analyze logical subtraction, two more mereological assumptions will be needed. See Section 8.

‘F’, ‘G’, and so on. Fourth, L contains the connectives ‘$\neg$’, ‘$\wedge$’, and ‘$\vee$’. Fifth, L contains the quantifiers ‘$\forall$’ and ‘$\exists$’.

The standard definitions from first-order logic apply to expressions of L. A ‘term’, for instance, is a constant or a variable. In addition, for each natural number n, for each n-place predicate $\mathcal{F}$, and for all terms $\tau_1, \ldots, \tau_n$, the expression $\mathcal{F}\tau_1 \ldots \tau_n$ is called an ‘atomic formula’. All other formulas are defined in the usual way. A ‘sentence’ is a formula with no free variables.[10]

The truth conditions, for sentences of L, invoke variable assignments and models. A variable assignment is a function which maps every variable in L to an object. A model consists of six ingredients: a set of states, a two-place parthood relation among states, a set of objects, a constant assignment, a function which maps atomic formulas of L to sets of states, and another function which maps atomic formulas of L to sets of states. Let us consider each of these in turn.

The first and second ingredients are, respectively, a set of states S and a corresponding parthood relation $\sqsubseteq$. Together, S and $\sqsubseteq$ jointly form a complete, well-founded lattice. So they satisfy the six conditions from Section 2.

The third ingredient is a set of objects I. Intuitively, the members of I are the objects which terms in L can be used to express. So every variable assignment maps each variable in L to an object in I. And similarly, the constants in L denote objects in I too.

The fourth ingredient is a constant assignment. A constant assignment is a function which maps each constant in L to an object in I. So constant assignments are the formal tools that specify the objects which constants denote.

The fifth and sixth ingredients are functions which map atomic formulas of L to sets of states. Let ‘$V\_$’ denote one of these functions and let ‘$F\_$’ denote the other. Intuitively, the set of states to which $V\_$ maps a given atomic formula contains exactly the states which make that formula true. And intuitively, the set of states to which $F\_$ maps a given atomic

[10]For a thorough presentation of these definitions, see (Enderton, 2001).

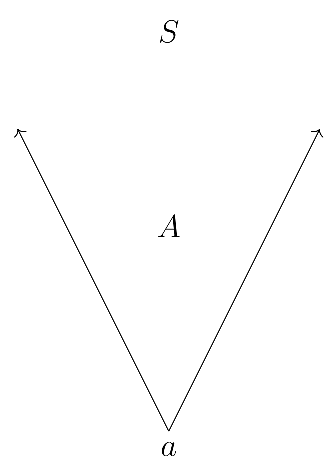
formula contains exactly the states which make that formula false.

Here is an example. Let 'P' be a one-place predicate which expresses the property of being green. Let 'a' be a constant which denotes grass. Then 'Pa' says that grass is green. So V_{Pa} is the set of all states in which grass is green: the state of grass being green, the state of grass being green and roses being red, and so on. And F_{Pa} is the set of all states in which grass is not green: the state of grass being not green, the state of grass being not green and roses being red, and so on.

The functions $V_{\_}$ and $F_{\_}$ have several features. Most of those features are pretty technical; so I relegate them to Appendix B. One of those features, however, has many philosophically interesting implications for the analyses to come. So it is worth discussing here.

To explain what that feature is, a preliminary definition will be useful. Let A be a subset of a given set S of states. Then A is a 'cone' just in case there exists a state a such that for all s in S, s is in A if and only if $a \sqsubseteq s$; in such a case, the state a is called the 'point' of A. In other words, a cone is a collection of states which all share a single state—the cone's point—in common.

The picture below illustrates the basic idea behind cones.



The entire space represents the set of states S. The vertex a, where the two lines meet, represents a particular state. The two lines, and the space inside them, represent all of the states in S which contain a as a part. So those two lines, along with the space inside them, represent the cone A whose point is a. The empty space surrounding the cone represents states which do not contain a as a part.

Now for the interesting feature of the functions $V_{\_}$ and $F_{\_}$: for each atomic formula in L, $V_{\_}$ maps that formula to a cone in S, and $F_{\_}$ maps that formula to a cone in S. For example, take the one-place predicate 'P' which expresses the property of being green, and take the constant 'a' which denotes grass. Then V_{Pa} is a cone: in particular, it is the cone whose point is the state of grass being green. And F_{Pa} is a cone too: in particular, it is the cone whose point is the state of grass not being green.

According to the analysis of content to come, the functions $V_{\_}$ and $F_{\_}$ play the role of assigning contents to atomic formulas. The cone V_{Pa}, according to that analysis, partially comprises the content of the formula 'Pa'; that cone is part of what that formula means. Similarly, the cone F_{Pa}, according to that analysis, also partially comprises the content of the formula 'Pa'; that cone is also part of what that formula means. And taken together, these two cones—V_{Pa} and F_{Pa}—comprise the content of the formula 'Pa'. They are what that formula means.

As I will explain later, cones are quite important. The truth conditions to come, along with a particular analysis of content, can be used to derive the following result: the content of any sentence in L whatsoever is determined by a series of cones. Basically, for each sentence in L, the content of that sentence is the union of cones which have certain nice features.

Here is a quick overview of how the truth conditions will work. Basically, those conditions describe—in a complete and general way—two different truth-theoretic relations between sentences and states. One is the relation of verification. Each sentence in L is verified—that is, made true—by certain states; in what follows, say that the 'verifiers' of a sentence are the states which make that sentence true. The other is the relation of falsification. Each sentence in L is falsified—that is, made false—by certain states too; in what follows, say that the 'falsifiers' of a sentence are the states which make that sentence false. The truth conditions describe how the verifiers and falsifiers of simpler sentences determine the verifiers and falsifiers of more complicated sentences. In this way, the truth conditions provide a complete theory of what makes first-order sentences true and what makes first-order sentences false.

The truth conditions rely on the models and variable assignments mentioned earlier. For brevity, in the presentation of truth conditions in this section, I omit some details: I omit all mention of variable assignments, for instance; and I omit all mention of constant assignments. For the fully rigorous formulation of the truth conditions, which includes those details, see Appendix B.

With all that as background, here are the truth conditions. Let S be a set of states, and let $\sqsubseteq$ be a partial order on S. Suppose that S and $\sqsubseteq$ jointly form a complete, well-founded lattice. Let I be a set of objects. And let $V\_$ and $F\_$ be the functions described above. Then verification and falsification is defined according to the six conditions below. To start, here are the truth conditions for atomic formulas.

Atomic

Let s be a state in S, and let $\mathcal{F}\tau_1 \ldots \tau_n$ be an atomic formula.

- Verification: s verifies $\mathcal{F}\tau_1 \ldots \tau_n$ if and only if s is in $V_{\mathcal{F}\tau_1 \ldots \tau_n}$.
- Falsification: s falsifies $\mathcal{F}\tau_1 \ldots \tau_n$ if and only if s is in $F_{\mathcal{F}\tau_1 \ldots \tau_n}$.[11]

Here are the truth conditions for negations of formulas.

Negation

Let s be a state in S, and let ϕ be a sentence.

- Verification: s verifies $\neg\phi$ if and only if s falsifies ϕ.
- Falsification: s falsifies $\neg\phi$ if and only if s verifies ϕ.

[11] This is one of the main places where I am eliding some important formal details. If any of the terms in the expression $\mathcal{F}\tau_1 \ldots \tau_n$ are variables, then it only makes sense to say that a certain state verifies or falsifies $\mathcal{F}\tau_1 \ldots \tau_n$ relative to a particular assignment of objects to those variables; that is, relative to a particular variable assignment. Again, the details of this are in Appendix B.

Here are the truth conditions for conjunctive formulas.

<u>Conjunction</u>

Let s be a state in S, and let ϕ and ψ be sentences.

- Verification: s verifies $\phi \wedge \psi$ if and only if there are states t and u in S such that $t \sqsubseteq s$, $u \sqsubseteq s$, t verifies ϕ, and u verifies ψ.
- Falsification: s falsifies $\phi \wedge \psi$ if and only if for some state t in S such that $t \sqsubseteq s$, either t falsifies ϕ or t falsifies ψ.

Here are the truth conditions for disjunctive formulas.

<u>Disjunction</u>

Let s be a state in S, and let ϕ and ψ be sentences.

- Verification: s verifies $\phi \vee \psi$ if and only if for some state t in S such that $t \sqsubseteq s$, either t verifies ϕ or t verifies ψ.
- Falsification: s falsifies $\phi \vee \psi$ if and only if there are states t and u in S such that $t \sqsubseteq s$, $u \sqsubseteq s$, t falsifies ϕ, and u falsifies ψ.

Here are the truth conditions for universally quantified formulas.

<u>Universal</u>

Let s be a state in S, let χ be a variable, and let $\phi(\chi)$ be a formula in which only χ appears free.

- Verification: s verifies $\forall\chi\phi(\chi)$ if and only if for each object o in I, there is a state t in S such that $t \sqsubseteq s$ and when χ is interpreted as denoting o, t verifies $\phi(\chi)$.
- Falsification: s falsifies $\forall\chi\phi(\chi)$ if and only if for some object o in I, there is a state

t in S such that $t \sqsubseteq s$ and when χ is interpreted as denoting o, t falsifies $\phi(\chi)$.

Here are the truth conditions for existentially quantified formulas.

Existential

Let s be a state in S, let χ be a variable, and let $\phi(\chi)$ be a formula in which only χ appears free.

- Verification: s verifies $\exists\chi\phi(\chi)$ if and only if for some object o in I, there is a state t in S such that $t \sqsubseteq s$ and when χ is interpreted as denoting o, t verifies $\phi(\chi)$.
- Falsification: s falsifies $\exists\chi\phi(\chi)$ if and only if for each object o in I, there is a state t in S such that $t \sqsubseteq s$ and when χ is interpreted as denoting o, t falsifies $\phi(\chi)$.

Together, these six conditions provide a complete theory of truth and falsity for sentence of first-order logic. These conditions, along with the Complete Lattice theory of states, comprise what I have been calling 'Statespace'.

For an example application of Statespace, consider the sentence '$Pa \wedge Qb$'. As before, suppose that 'P' represents the property of being green, and suppose that 'a' denotes grass. In addition, suppose that 'Q' represents the property of being red, and suppose that 'b' denotes roses. Let r_1 be the state of grass being green, and let r_2 be the state of roses being red. Then by Existence of Least Upper Bound, the condition from Section 2, there exists a least upper bound of r_1 and r_2; namely, $r_1 \sqcap r_2$. Now, according to the above condition Conjunction, the state $r_1 \sqcap r_2$ verifies '$Pa \wedge Qb$' if and only if for some $t \sqsubseteq r_1 \sqcap r_2$ and some $u \sqsubseteq r_1 \sqcap r_2$, t verifies 'Pa' and u verifies 'Qb'. In other words, $r_1 \sqcap r_2$ verifies the sentence "Grass is green and roses are red" just in case $r_1 \sqcap r_2$ contains a part which verifies "Grass is green" and a part which verifies "Roses are red." And that is indeed the case. To see why, for starters, note that $r_1 \sqcap r_2$ contains both the state r_1 and the state r_2 as parts: this follows from the definition of least upper bound. In addition, the above condition Atomic implies that r_1 verifies 'Pa', since the state of grass being green is in the cone V_{Pa}. Atomic also implies that r_2 verifies 'Qb', since the state of roses being red is in the cone V_{Qb}. Therefore, the state of grass being green and roses being red – that is, $r_1 \sqcap r_2$ – verifies the sentence "Grass is green and roses are red" – that is, '$Pa \wedge Qb$'.

Statespace is similar to standard theories of situation semantics (Barwise, 1981; Barwise & Perry, 1983; Kratzer, 1989; 2012). But there are some important differences. For example, Kratzer's theory does not assume the existence of greatest lower bounds or least upper bounds. And the truth conditions in Kratzer's theory do not invoke parts of states; not, that is, in the direct way that the truth conditions in Statespace do.

Actually, as discussed in Section 9, this latter difference is one of the most unique aspects of Statespace. According to Statespace, any given sentence is verified or falsified by a state just in case, very roughly put, the *parts* of that sentence are verified or falsified by the *parts* of that state. According to most all other theories of states and truth, however, certain kinds of sentences are verified or falsified by a state just in case, very roughly put, the *parts* of those sentences are verified or falsified by *that state itself*, rather than that state's parts. And this difference between Statespace and those other theories—which I discuss in much more detail in Section 9—turns out to be extremely important. For it is what allows Statespace, along with a particular analysis of content, to imply that the contents of sentences in L are determined by cones. So this difference is responsible for the simple, clear, and elegant account of content which Statespace supports.

Universal and Existential differ from the truth conditions, for quantifies, endorsed by other theories of states and truth in the literature (Elgin, 2021, p. 8; Fine, 2017c, p. 568). Other truth conditions invoke totality states, that is, states which say exactly what individuals exist. Roughly put, according to those other conditions, a verifier of a universal is a fusion of (i) states which verify that universal's instances, along with (ii) a totality state which says that the objects in those instances are all the object that there are. Similarly for falsifiers of universals, and for verifiers and falsifiers of existentials.

Because they contain totality states as parts, each verifier s of a universal—according to those other truth conditions—necessitates the truth of that universal. For each such s contains (i) verifiers of that universal's instances, and (ii) a state which says that the objects in those instances are all the objects in existence. By (i), each instance of that universal is verified by a part of s. By (ii), there are no other instances of that universal apart from the instances covered in (i). So that universal must be true, if s obtains.

I dislike this feature of totality states; so I do not include them in Universal and Existential. Quantificational sentences in first-order logic can be made true, or made false, without being made necessarily true or necessarily false. In my view, the main reason to accept that verifiers necessitate what they verify—and that falsifiers necessitate what they falsify—is metaphysical: it is based on a particular conception of what sorts of things states are, and how their mere existences make truths and falsehoods hold necessarily (Armstrong, 2004). And as mentioned in Section 2, in this paper, I do not adopt any particular metaphysical view of states. I am interested in characterizing the semantic relations of verification and falsification, and in using that characterization to analyze a host of notions related to meaning. And from the perspective of that project, it is at best unrequired—and at worst misguided—to adopt posits about relations of metaphysical necessitation between states and sentences of first-order logic.[12]

Nevertheless, totality states can be easily added to the truth conditions above. Just change Universal so that it requires verifiers and falsifiers of universals to contain totality states as parts, and change Existential similarly. The reader is welcome to adopt this char-

[12]Here is another argument for thinking that totality states should not figure in the verification and falsification conditions for quantifiers. Insisting that verifiers necessitate universals is like insisting that set-theoretic models—from the standard account of the truth conditions of sentences in first-order logic, which is based on set theory—necessitate universals. But that would be bizarre. It would be bizarre if those models necessitated the truths of quantificational sentences, perhaps due to an "And that's all" totality clause in their set-theoretic truth conditions. Model theorists, in the early days of model theory, did not adopt any clauses like that in their semantics. And they were right not to, since they were engaged in the semantical project of providing truth conditions, not in the metaphysical project of capturing relations of metaphysical necessitation. Similarly, we should not adopt any clauses about metaphysically necessitating totalities in our theories of states and truth, since those theories are engaged only in the semantical project of characterizing relations of verification and falsification.

acterization of those truth conditions, if they prefer it.

Despite its relative simplicity, Statespace is extremely powerful. It can be used to analyze many different notions related to meanings. In what follows, I focus on thirteen in particular. Then I compare Statespace to other theories of how states verify, and falsify, sentences in first-order logic; as will become clear, there are several reasons for preferring Statespace over those other theories.

4 Contents, Propositions, and Parts

In this section, I use Statespace to analyze the contents of first-order sentences. On the basis of that analysis, I propose an analysis of what propositions are. Then I propose analyses of three different kinds of partial content. Along the way, I discuss some attractive, interesting, and elegant consequences of the conditions in the previous sections.

By way of preparation, here are definitions of two different kinds of sets. First, for each sentence ϕ in L, let V_ϕ be the set of all states which verify ϕ. Second, for each sentence ϕ in L, let F_ϕ be the set of all states which falsify ϕ.[13] So intuitively, V_ϕ contains all and only the states which make ϕ true, and F_ϕ contains all and only the states which make ϕ false.

It turns out that for each sentence ϕ, the sets V_ϕ and F_ϕ have an extremely nice feature. In order to explain what that feature is, however, another definition will be needed: the definition describes a certain way in which a cone, in S, may be 'as big as possible'. So let S and $\sqsubseteq$ jointly form a complete, well-founded lattice. Let A be a subset of S, and let C be a subset of A. Then C is a 'maximal cone in A' just in case the following conditions hold.

1. C is a cone.

2. For every cone C' which is a subset of A and which contains C, C is identical to C'.

In other words, a cone C is maximal in A just in case there is no cone which is both (i) a

[13] Of course, V_ϕ and F_ϕ are defined relative to some particular model; that is, relative to some particular choice of a set of states S, a partial order $\sqsubseteq$, and so on. In this section, and in the sections to come, I suppress most mention of the background models which supply the verifiers and falsifiers for the sentences in question.

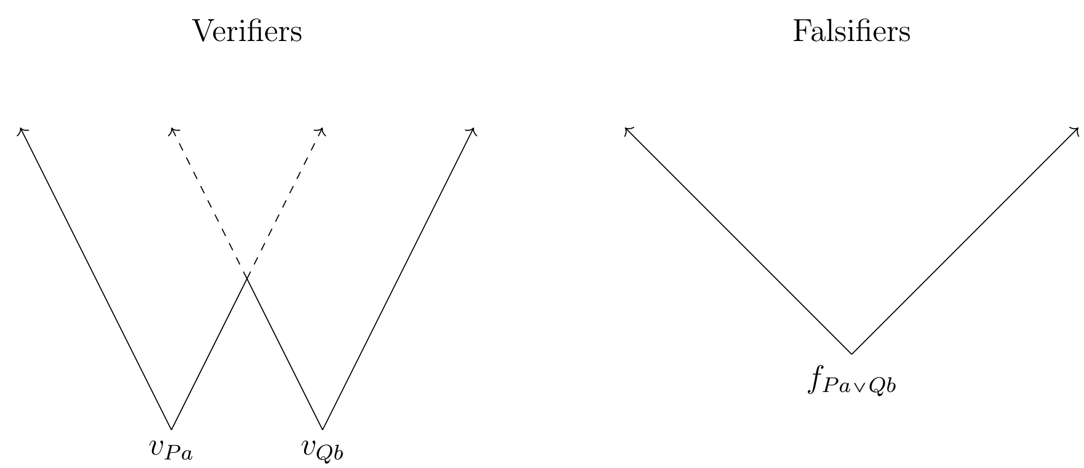
subset of A, and (ii) strictly larger than C, in the sense that it contains C as a proper subset. So a cone is maximal, in a subset, just in case there is no way to make that cone bigger, while still keeping it in that subset. The cone is, in that sense, 'as big as possible'.

For example, recall the sentence 'Pa', where 'P' expresses the property of being green and 'a' denotes grass. As mentioned in Section 3, both V_{Pa} and F_{Pa} are cones. As a simple proof shows, V_{Pa} is a maximal cone in V_{Pa}, and F_{Pa} is a maximal cone in F_{Pa}. That is, each of V_{Pa} and F_{Pa} is a maximal cone in itself.

Now for the extremely nice feature which every set V_ϕ, and every set F_ϕ, has.

Theorem 1. *For each sentence ϕ, V_ϕ is a union of cones which are maximal in V_ϕ, and F_ϕ is a union of cones which are maximal in F_ϕ.*

The fully rigorous derivation of this result is quite complicated; see theorem C.2 in Appendix C for the proof. But the basic idea can be illustrated by a simple example. As before, let 'Pa' be a sentence which says that grass is green, and let 'Qb' be a sentence which says that roses are red. Here is a visual representation of the verifiers, and falsifiers, of the sentence '$Pa \vee Qb$'.



The area enclosed by the solid lines on the left represents $V_{Pa \vee Qb}$. The area enclosed by the solid lines on the right represents $F_{Pa \vee Qb}$. Both areas can be represented as unions of cones which are maximal in $V_{Pa \vee Qb}$ and $F_{Pa \vee Qb}$, respectively. For instance, the area which represents $V_{Pa \vee Qb}$ is the union of two cones which are maximal in $V_{Pa \vee Qb}$. One cone, whose point is v_{Pa}, contains the states in which grass is green. The other cone, whose point is v_{Qb}, contains the states in which roses are red. Together, their union is the collection of all states in which grass is green or roses are red: namely, $V_{Pa \vee Qb}$.[14] Similarly, the area which represents $F_{Pa \vee Qb}$ is the union of a single cone which is maximal in $F_{Pa \vee Qb}$. The point of this cone is the state of grass not being green and roses not being red. And its union—or more simply, the cone itself—is $F_{Pa \vee Qb}$.[15]

Theorem 1 is, in many ways, the keystone of my approach to states, truth, and meaning for sentences in first-order logic. As shown in the rest of this section, that theorem supports simple and attractive analyses of content, propositions, and partial content. And as shown in later sections, that theorem supports many other simple and attractive analyses, including analyses of exact verification, exact falsification, subject matter, entailment, logical subtraction, and more. So it is hard to overstate the importance of this theorem. It provides the foundation for all that is to come.

Now for the analysis of content. Basically, it says that the content of a first-order sentence consists of the verifiers and falsifiers of that sentence.

Content

Let ϕ be a sentence in L. The *content* of ϕ is the pair of sets $\langle V_\phi, F_\phi \rangle$.

In other words, the content of a sentence is a pair containing (i) that sentence's verifiers, and (ii) that sentence's falsifiers. Say that V_ϕ is the 'positive content' of ϕ, and say that F_ϕ is the

[14] Each dashed line represents that portion of a maximal cone's boundary which is contained in another maximal cone. The dashed line pointing upwards and to the left from v_{Qb}, for instance, represents the portion of that maximal cone's boundary—the cone whose point is v_{Qb}—which is contained in the maximal cone whose point is v_{Pa}.

[15] A straightforward but technical exercise shows that this picture is, in general, an accurate depiction of the structure of the sets $V_{Pa \vee Qb}$ and $F_{Pa \vee Qb}$. In particular, it can be shown that so long as V_{Pa} is not a subset of V_{Qb} and V_{Qb} is not a subset of V_{Pa}, $V_{Pa \vee Qb}$ contains exactly two distinct cones which are maximal in $V_{Pa \vee Qb}$. And it can be shown that $F_{Pa \vee Qb}$ always contains exactly one cone which is maximal in $F_{Pa \vee Qb}$.

'negative content' of ϕ.

For example, take the sentence "Grass is green": namely, 'Pa'. V_{Pa} is the set of all states which make that sentence true. F_{Pa} is the set of all states which make that sentence false. And together, the pair $\langle V_{Pa}, F_{Pa} \rangle$ is the content of 'Pa'; that pair is the content of the sentence "Grass is green." V_{Pa} is the positive content of that sentence, and F_{Pa} is the negative content of that sentence.

There is much to like about Content. For starters, it is intuitively plausible. It says that the content of a sentence consists of all the different ways of (i) making that sentence true, and (ii) making that sentence false. To put it another way: if a worldly state fails to determine the truth value of a sentence, then that state does not contribute to that sentence's content. And that is, of course, extremely plausible. It makes sense to claim that if something has no effect on whether a sentence is true or false, then that something is not in the content of that sentence.

In addition, Content implies that classically logically equivalent formulas often have distinct contents. For example, consider the sentence '$Pa \vee \neg Pa$' which says that either grass is green or grass is not green. And consider the sentence '$Qb \vee \neg Qb$' which says that either roses are red or roses are not red. Given classical truth conditions, these sentences are logically equivalent. And given Content, the contents of these sentences are distinct. For some verifiers of '$Pa \vee \neg Pa$' neither verify nor falsify '$Qb \vee \neg Qb$'. The state of grass being green, for example, verifies the former but neither verifies nor falsifies the latter. So these sentences have different contents.

That is, intuitively, the right result. The sentence '$Pa \vee \neg Pa$' is about grass being green or not green. It is not about roses at all. And the sentence '$Qb \vee \neg Qb$' is about roses being red or not red. It is not about grass at all. These sentences are clearly about different things. They have different meanings. And Content respects all that.

As a simple proof shows, Content implies that 'Pa' and '$Pa \vee (Pa \wedge Qb)$' have the same content. One might take that to be problematic. This identification is, after all, often rejected in the literature on states, verification, and falsification. For when combined with two assumptions about counterfactuals—one called 'Simplification' and one called 'Substitution'—this identification has unintuitive results (Deigan, 2020, p. 525; Fine, 2017c, p. 571). So one might conclude that this identification should be rejected.

A full response to this objection will have to wait until Section 7. In that section, I use Statespace to formulate truth conditions for counterfactuals. Those conditions, along with the account of entailment given in Section 6, imply that Simplification is false. That result, moreover, is quite well-motivated: many standard semantics for counterfactuals imply the falsity of Simplification. So ultimately, it is perfectly fine that 'Pa' and '$Pa \vee (Pa \wedge Qb)$' have the same content.

Now for the analysis of propositions. Basically, the analysis says that propositions are pairs of sets which are, themselves, unions of maximal cones.

Proposition

$\mathcal{P}$ is a *proposition* if and only if $\mathcal{P}$ is a pair $\langle \mathcal{P}_V, \mathcal{P}_F \rangle$ such that

(i) $\mathcal{P}_V$ and $\mathcal{P}_F$ are sets of states,

(ii) $\mathcal{P}_V$ is a union of cones which are maximal in $\mathcal{P}_V$, and

(iii) $\mathcal{P}_F$ is a union of cones which are maximal in $\mathcal{P}_F$.

In other words, propositions are pairs, the elements of which are unions of maximal cones. For each proposition $\mathcal{P} = \langle \mathcal{P}_V, \mathcal{P}_F \rangle$, a state makes that proposition true just in case that state is in $\mathcal{P}_V$, and that state makes that proposition false just in case that state is in $\mathcal{P}_F$. In what follows, call the states in $\mathcal{P}_V$ the 'verifiers' of $\mathcal{P}$, and call the states in $\mathcal{P}_F$ the 'falsifiers' of $\mathcal{P}$. In addition, say that $\mathcal{P}_V$ is the 'positive content' of $\mathcal{P}$, and say that $\mathcal{P}_F$ is the 'negative content' of $\mathcal{P}$.

Proposition is an extremely attractive analysis of propositions. Along with Content, it implies that the contents of sentences are propositions; that is, obviously, an attractive result. It also allows for two propositions to hold at exactly the same worlds, and yet not be identical; for reasons similar to those given above, in the discussion of the contents of '$Pa \vee \neg Pa$' and '$Qb \vee \neg Qb$', that is an attractive result too.

There are several different kinds of parthood relations among propositions, and therefore, among the contents of sentences. In this section, I analyze three of them. The analyses are based on the ways in which (i) the maximal cones that comprise one proposition, may relate to (ii) the maximal cones that comprise another proposition. And together, they capture three different ways in which the content of one sentence may be part of the content of another sentence.[16]

Before continuing, it is worth making an observation about the points of maximal cones. Suppose that a cone is maximal in the set of verifiers V_ϕ of a sentence ϕ. Then the point of that cone is one of the *smallest* states which makes ϕ true, in the sense that it makes ϕ true but no proper part of it does. Similarly, suppose that a cone is maximal in the set of falsifiers F_ϕ of a sentence ϕ. Then the point of that cone is one of the *smallest* states which makes ϕ false, in the sense that it makes ϕ false but no proper part of it does. In other words, to put it roughly but perhaps more intuitively, the points of maximal cones in sets of verifiers – or falsifiers – are themselves the smallest verifiers – or falsifiers – that there are.

Now for the first two parthood conditions. The first says what it takes for one proposition's positive content to be part of another proposition's positive content. The second says what it takes for one proposition's negative content to be part of another proposition's negative content.

Positive Partial Content

Let $\mathcal{P} = \langle \mathcal{P}_V, \mathcal{P}_F \rangle$ and $\mathcal{Q} = \langle \mathcal{Q}_V, \mathcal{Q}_F \rangle$ be propositions. Then the positive content of $\mathcal{P}$ *is part of* the positive content of $\mathcal{Q}$ if and only if the following two conditions hold.

[16] In this paper, I restrict my discussion to parthood relations among states and among propositions. I do not discuss parthood relations among related—though importantly distinct—items, such as the attitudinal objects discussed in (Moltmann, 2017).

(i) For every cone $C_{\mathcal{P}}$ which is maximal in $\mathcal{P}_V$ and which has point $c_{\mathcal{P}}$, there is a cone $C_{\mathcal{Q}}$ such that (a) $C_{\mathcal{Q}}$ is maximal in $\mathcal{Q}_V$, and (b) $C_{\mathcal{Q}}$ has a point $c_{\mathcal{Q}}$ such that $c_{\mathcal{P}} \sqsubseteq c_{\mathcal{Q}}$.

(ii) For every cone $C_{\mathcal{Q}}$ which is maximal in $\mathcal{Q}_V$ and which has point $c_{\mathcal{Q}}$, there is a cone $C_{\mathcal{P}}$ such that (a) $C_{\mathcal{P}}$ is maximal in $\mathcal{P}_V$, and (b) $C_{\mathcal{P}}$ has a point $c_{\mathcal{P}}$ such that $c_{\mathcal{P}} \sqsubseteq c_{\mathcal{Q}}$.

Negative Partial Content

Let $\mathcal{P} = \langle \mathcal{P}_V, \mathcal{P}_F \rangle$ and $\mathcal{Q} = \langle \mathcal{Q}_V, \mathcal{Q}_F \rangle$ be propositions. Then the negative content of $\mathcal{P}$ *is part of* the negative content of $\mathcal{Q}$ if and only if the following two conditions hold.

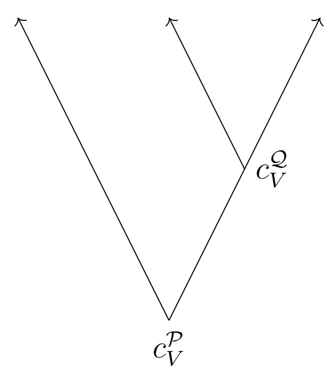
(i) For every cone $C_{\mathcal{P}}$ which is maximal in $\mathcal{P}_F$ and which has point $c_{\mathcal{P}}$, there is a cone $C_{\mathcal{Q}}$ such that (a) $C_{\mathcal{Q}}$ is maximal in $\mathcal{Q}_F$, and (b) $C_{\mathcal{Q}}$ has a point $c_{\mathcal{Q}}$ such that $c_{\mathcal{P}} \sqsubseteq c_{\mathcal{Q}}$.

(ii) For every cone $C_{\mathcal{Q}}$ which is maximal in $\mathcal{Q}_F$ and which has point $c_{\mathcal{Q}}$, there is a cone $C_{\mathcal{P}}$ such that (a) $C_{\mathcal{P}}$ is maximal in $\mathcal{P}_F$, and (b) $C_{\mathcal{P}}$ has a point $c_{\mathcal{P}}$ such that $c_{\mathcal{P}} \sqsubseteq c_{\mathcal{Q}}$.

Roughly put, Positive Partial Content says that the positive content of $\mathcal{P}$ is part of the positive content of $\mathcal{Q}$ just in case (i) every smallest verifier of $\mathcal{P}$ is part of some smallest verifier of $\mathcal{Q}$, and (ii) every smallest verifier of $\mathcal{Q}$ contains some smallest verifier of $\mathcal{P}$ as a part. And roughly put, Negative Partial Content says that the negative content of $\mathcal{P}$ is part of the negative content of $\mathcal{Q}$ just in case (i) every smallest falsifier of $\mathcal{P}$ is part of some smallest falsifier of $\mathcal{Q}$, and (ii) every smallest falsifier of $\mathcal{Q}$ contains some smallest falsifier of $\mathcal{P}$ as a part.

The following terminology will be helpful. The positive content of sentence ϕ 'is part of' the positive content of sentence ψ just in case (i) ϕ expresses some proposition $\mathcal{P}_\phi$, (ii) ψ expresses some proposition $\mathcal{P}_\psi$, and (iii) the positive content of $\mathcal{P}_\phi$ is part of the positive content of $\mathcal{P}_\psi$. Similarly, the negative content of sentence ϕ 'is part of' the negative content of sentence ψ just in case (i) ϕ expresses some proposition $\mathcal{P}_\phi$, (ii) ψ expresses some proposition $\mathcal{P}_\psi$, and (iii) the negative content of $\mathcal{P}_\phi$ is part of the negative content of $\mathcal{P}_\psi$.

The picture below will help illustrate the general idea behind Positive Partial Content; an entirely analogous picture would illustrate the general idea behind Negative Partial Content too. Let $\mathcal{P} = \langle C_V^{\mathcal{P}}, \mathcal{P}_F \rangle$ and $\mathcal{Q} = \langle C_V^{\mathcal{Q}}, \mathcal{Q}_F \rangle$ be propositions such that $C_V^{\mathcal{P}}$ is a cone with point $c_V^{\mathcal{P}}$ and $C_V^{\mathcal{Q}}$ is a cone with point $c_V^{\mathcal{Q}}$. Suppose that according to Positive Partial Content, the positive content of $\mathcal{P}$ is part of the positive content of $\mathcal{Q}$; in the present example, this is equivalent to supposing that $c_V^{\mathcal{P}} \sqsubseteq c_V^{\mathcal{Q}}$. Then the following picture illustrates how the verifiers of $\mathcal{P}$ and the verifiers of $\mathcal{Q}$ relate to one another.



The lines beginning at $c_V^{\mathcal{P}}$, and the space between them, jointly represent the cone $C_V^{\mathcal{P}}$. The lines beginning at $c_V^{\mathcal{Q}}$, and the space between them, jointly represent the cone $C_V^{\mathcal{Q}}$. And the line segment running from $c_V^{\mathcal{P}}$ to $c_V^{\mathcal{Q}}$ represents something else as well: it represents the fact that $c_V^{\mathcal{Q}}$ contains $c_V^{\mathcal{P}}$ as a part. Moreover, as a simple formal exercise shows, it follows that every state in $C_V^{\mathcal{Q}}$ contains $c_V^{\mathcal{P}}$ as a part; that, too, is represented by the line segment running from $c_V^{\mathcal{P}}$ to the cone $C_V^{\mathcal{Q}}$.

Positive Partial Content and Negative Partial Content have many attractive implications. For example, intuitively, the positive content of 'Pa' is part of the positive content of '$Pa \wedge Qb$'. And as a simple argument shows, Positive Partial Content gets that right. Similarly, intuitively, the negative content of '$Pa \vee Qb$' is part of the negative content of 'Pa'. NEGATIVE PARTIAL CONTENT gets that right. And intuitively, 'Pa' is neither part of the positive content, nor part of the negative content, of '$Pa \vee Qb$'. POSITIVE PARTIAL CONTENT and NEGATIVE PARTIAL CONTENT imply that too.[17]

Now for the third parthood condition. Basically, this condition says what it takes for one proposition's content—rather than just its positive content or negative content specifically—be part of another proposition's content.

PARTIAL CONTENT

Let $\mathcal{P}$ and $\mathcal{Q}$ be propositions. Then $\mathcal{P}$ *is part of* $\mathcal{Q}$ if and only if

(i) the positive content of $\mathcal{P}$ is part of the positive content of $\mathcal{Q}$, and

(ii) the negative content of $\mathcal{P}$ is part of the negative content of $\mathcal{Q}$.

In other words, one proposition is part of another just in case the positive and negative contents of the former are parts of the positive and negative contents of the latter, respectively.

PARTIAL CONTENT is an intuitively plausible analysis of parthood among propositions. It makes sense to say that one proposition only counts as part—completely part—of another just in case the former's positive content *and* negative content are parts of the latter's positive content *and* negative content, respectively. It makes sense to say, in other words, that one proposition is part of another just in case the above correspondences obtain between the points of the maximal cones that comprise the former and the points of the maximal cones that comprise the latter.

5 Exact Verification, Exact Falsification, Subject Matter, and Aboutness

In this section, I use Statespace to analyze exact verification, exact falsification, subject matter, and the aboutness relation. Along the way, I discuss some more nice consequences

[17] Consequently, these accounts of partial content satisfy the desiderata outlined in (Gemes, 1994; 1997).

of the conditions in the previous sections.

To start, here is the analysis of exact verification.

Exact Verification

Let ϕ be a sentence in L, and let s be a state. Then s *exactly verifies* ϕ if and only if

(i) s verifies ϕ, and

(ii) for all states r such that $r \subsetneq s$, r does not verify ϕ.

In other words, the exact verifiers of a sentence are the smallest states which verify that sentence. And here is the analysis of exact falsification.

Exact Falsification

Let ϕ be a sentence in L, and let s be a state. Then s *exactly falsifies* ϕ if and only if

(i) s falsifies ϕ, and

(ii) for all states r such that $r \subsetneq s$, r does not falsify ϕ.

In other words, the exact falsifiers of a sentence are the smallest states which falsify that sentence.

There is debate, in the literature, over whether theories of states and truth can (i) use the relations of verification and falsification to analyze the relations of exact verification and exact falsification, or (ii) use the relations of exact verification and exact falsification to analyze the relations of verification and falsification (Deigan, 2020, p. 524; Fine, 2017c, p. 565). As Exact Verification and Exact Falsification show, (i) is perfectly viable. The exact verifiers of a sentence can be analyzed in terms of verification more generally, and the exact falsifiers of a sentence can be analyzed in terms of falsification more generally.[18]

[18]This is not to say, of course, that (ii) cannot be done as well. I suspect, actually, that both (i) and (ii) can be done. That is, I suspect that verification and falsification on the one hand, and exact verification and exact falsification on the other, are inter-analyzable: either pair of relations, when posited primitively, could be used to analyze the others.

Now for subject matter. By way of preparation, for each sentence ϕ, and for each cone C which is maximal in V_ϕ, let v_C be the point of C. Let v_ϕ be the least upper bound of all these v_C. Similarly, for each sentence ϕ, and for each cone C which is maximal in F_ϕ, let f_C be the point of C. Let f_ϕ be the least upper bound of all these f_C. Then subject matter is analyzed as follows.

Subject Matter

Let ϕ be a sentence in L. The *subject matter* of ϕ is the pair $\langle v_\phi, f_\phi \rangle$.

In other words, the subject matter of a sentence is a pair consisting of (i) the fusion of all the different smallest ways of making that sentence true, and (ii) the fusion of all the different smallest ways of making that sentence false. In what follows, call v_ϕ the 'positive subject matter' of ϕ, and call f_ϕ the 'negative subject matter' of ϕ.

Subject Matter has several attractive implications. For instance, it implies that the subject matters of sentences play a role in determining those sentences' truth values. In particular, given Subject Matter, the positive subject matter of a sentence verifies that sentence, and the negative subject matter of a sentence falsifies that sentence. That is, as proved in Appendix D, the following theorem holds.

Theorem 2. *Let ϕ be a sentence with subject matter $\langle v_\phi, f_\phi \rangle$. Then v_ϕ is in V_ϕ and f_ϕ is in F_ϕ.*

And this is the intuitively correct result. For intuitively, the positive subject matter of a sentence should make that sentence true, and the negative subject matter of a sentence should make that sentence false. The subject matter of a sentence, in other words, should not be utterly unrelated to that sentence's truth value. And Subject Matter is attractive, insofar as it implies exactly that.

Finally, here is the analysis of the aboutness relation: the relation, that is, which obtains between sentences and what they are about.

Aboutness

Let ϕ be a sentence in L. Then ϕ is *about* its subject matter $\langle v_\phi, f_\phi \rangle$.

In other words, sentences are about their subject matters. In what follows, I will say that ϕ is 'positively about' v_ϕ, and I will say that ϕ is 'negatively about' f_ϕ.

Unlike other theories of the aboutness relation (Yablo, 2014, p. 43), Aboutness implies that a sentence and its negation are about different things. The sentence "Grass is green" is positively about grass being green, for instance, and the sentence "Grass is not green" is positively about grass not being green. That is a feature of Aboutness, not a bug. It is unintuitive, I think, to claim that sentences and their negations are about exactly the same things. Intuitively, if a sentence is about some particular way for things to be, then the negation of that sentence is about things not being that way. So intuitively, sentences and their negations are about different things. Aboutness is attractive, since it respects that.

6 Entailment and Containment

In this section, I analyze two more relations among propositions: the relation of entailment, and the relation of containment. As will become clear, there is a precise sense in which these analyses are complete. Roughly put, they exhaust all four of the different ways in which sets of verifiers and falsifiers might be among other sets of verifiers and falsifiers, respectively.

Here is the analysis of entailment. Basically, it says that one proposition entails another just in case (i) every way of making the first true is a way of making the second true, and

(ii) every way of making the second false is a way of making the first false.

Entailment

Let $\mathcal{P} = \langle \mathcal{P}_V, \mathcal{P}_F \rangle$ and $\mathcal{Q} = \langle \mathcal{Q}_V, \mathcal{Q}_F \rangle$ be propositions. Then $\mathcal{P}$ *entails* $\mathcal{Q}$ if and only if

(i) for all states s in $\mathcal{P}_V$ there is a state r in $\mathcal{Q}_V$ such that $r \sqsubseteq s$, and

(ii) for all states s in $\mathcal{Q}_F$ there is a state r in $\mathcal{P}_F$ such that $r \sqsubseteq s$.

In other words, equivalently, $\mathcal{P}$ entails $\mathcal{Q}$ if and only if $\mathcal{P}_V$ is a subset of $\mathcal{Q}_V$ and $\mathcal{Q}_F$ is a subset of $\mathcal{P}_F$.[19] Note that in what follows, I adopt the following terminology: sentence ϕ 'entails' sentence ψ just in case the content of ϕ entails the content of ψ.

For example, let ϕ and ψ be sentences in L. Then $\phi \wedge \psi$ entails ϕ. For by the truth condition Conjunction, given any state s in $V_{\phi \wedge \psi}$, there is a state r in V_ϕ such that $r \sqsubseteq s$. The truth condition Conjunction also implies that given any state s in F_ϕ, there is a state r in $F_{\phi \wedge \psi}$ such that $r \sqsubseteq s$: just let r be s itself. So by Entailment, $\phi \wedge \psi$ entails ϕ.

Entailment does a good job of capturing the intuitive notion of entailment. To see why, note that intuitively, if one sentence entails another then the truth of the former guarantees the truth of the latter. Entailment respects that: for according to (i) in Entailment, if one sentence entails another then the truth of the former guarantees the truth of the latter, insofar as anything which makes the former true must have a part which makes the latter true too. Similarly, note that intuitively, if one sentence entails another then the falsity of the latter guarantees the falsity of the former. Entailment respects that as well: for according to (ii) in Entailment, if one sentence entails another then the falsity of the latter guarantees the falsity of the former, insofar as anything which makes the latter false must have a part which makes the former false too.

Now for the analysis of containment. Basically, the analysis says that one proposition is contained in another just in case (i) every way of making the second true is a way of making

[19]For the key result in the simple proof that this biconditional is equivalent to the biconditional in Entailment, see theorem C.1 in Appendix C.

the first true, and (ii) every way of making the second false is a way of making the first false.

Containment

Let $\mathcal{P} = \langle \mathcal{P}_V, \mathcal{P}_F \rangle$ and $\mathcal{Q} = \langle \mathcal{Q}_V, \mathcal{Q}_F \rangle$ be propositions. Then $\mathcal{P}$ *is contained in* $\mathcal{Q}$ if and only if

(i) for all states s in $\mathcal{Q}_V$ there is a state r in $\mathcal{P}_V$ such that $r \sqsubseteq s$, and

(ii) for all states s in $\mathcal{Q}_F$ there is a state r in $\mathcal{P}_F$ such that $r \sqsubseteq s$.

In other words, equivalently, $\mathcal{P}$ is contained in $\mathcal{Q}$ if and only if $\mathcal{Q}_V$ is a subset of $\mathcal{P}_V$ and $\mathcal{Q}_F$ is a subset of $\mathcal{P}_F$.[20] Note that in what follows, I adopt the following terminology: sentence ϕ 'is contained in' sentence ψ just in case the content of ϕ is contained in the content of ψ.

The containment relation, as described by Containment, is connected to the parthood relation described by Partial Content. In particular, if Partial Content implies that proposition $\mathcal{P}$ is part of proposition $\mathcal{Q}$, then Containment implies that $\mathcal{P}$ is contained in $\mathcal{Q}$ too.[21] The reverse, however, does not hold: there are propositions $\mathcal{P}$ and $\mathcal{Q}$ such that (i) Containment implies that $\mathcal{P}$ is contained in $\mathcal{Q}$, and yet (ii) Partial Content implies that $\mathcal{P}$ is not part of $\mathcal{Q}$.[22]

Together, Entailment and Containment support an elegant classification of four different subset relations among verifiers, and among falsifiers, of different propositions. To understand exactly how, take any two propositions $\mathcal{P} = \langle \mathcal{P}_V, \mathcal{P}_F \rangle$ and $\mathcal{Q} = \langle \mathcal{Q}_V, \mathcal{Q}_F \rangle$. And consider the following four different ways in which (i) one of these propositions' verifiers might be among the other propositions' verifiers, and (ii) one of these propositions' falsifiers might be among the other propositions' falsifiers.

(1) $\mathcal{P}_V \subseteq \mathcal{Q}_V$ and $\mathcal{Q}_F \subseteq \mathcal{P}_F$.

(2) $\mathcal{Q}_V \subseteq \mathcal{P}_V$ and $\mathcal{Q}_F \subseteq \mathcal{P}_F$.

[20]Theorem C.1, in Appendix C, is the key result which supports the proof that this biconditional is equivalent to the biconditional in Containment.

[21]For the proof, see theorem E.3 in Appendix E.

[22]For the proof, see theorem E.4 in Appendix E.

(3) $\mathcal{Q}_V \subseteq \mathcal{P}_V$ and $\mathcal{P}_F \subseteq \mathcal{Q}_F$.

(4) $\mathcal{P}_V \subseteq \mathcal{Q}_V$ and $\mathcal{P}_F \subseteq \mathcal{Q}_F$.

As a simple proof shows, Entailment and Containment imply the following: each of (1)–(4) corresponds to an instance of the entailment relation or an instance of the containment relation. Here is how.

(1) $\mathcal{P}_V \subseteq \mathcal{Q}_V$ and $\mathcal{Q}_F \subseteq \mathcal{P}_F$ if and only if $\mathcal{P}$ entails $\mathcal{Q}$.

(2) $\mathcal{Q}_V \subseteq \mathcal{P}_V$ and $\mathcal{Q}_F \subseteq \mathcal{P}_F$ if and only if $\mathcal{P}$ is contained in $\mathcal{Q}$.

(3) $\mathcal{Q}_V \subseteq \mathcal{P}_V$ and $\mathcal{P}_F \subseteq \mathcal{Q}_F$ if and only if $\mathcal{Q}$ entails $\mathcal{P}$.

(4) $\mathcal{P}_V \subseteq \mathcal{Q}_V$ and $\mathcal{P}_F \subseteq \mathcal{Q}_F$ if and only if $\mathcal{Q}$ is contained in $\mathcal{P}$.

So Entailment and Containment are, in a sense, exhaustive. There are four combinations of ways in which the verifiers and falsifiers of one proposition might be among, or might include, the verifiers and falsifiers of another proposition, respectively: namely, the ways listed in (1)–(4) above. Each of those four ways corresponds to a distinct instance of the entailment relation, or the containment relation, obtaining between the propositions in question.

This exhaustive correspondence is, I think, an extremely attractive feature of the present approach to propositions, entailment, and containment. The relations of entailment and containment capture the four different ways in which one proposition's verifiers and falsifiers might be among, or might include, another proposition's verifiers and falsifiers, respectively. And that is a nice consequence of the present approach to meaning in first-order logic.

7 Counterfactuals

Statespace provides a basis from which to formulate truth conditions for counterfactuals. In this section, I present those conditions. Then I use them to defend Statespace—and also Content—against an objection mentioned earlier. Finally, I briefly compare those conditions to other accounts of the semantics of counterfactuals.

The truth conditions rely on a 'comparative similarity' relation among states. Basically, this relation captures context-dependent facts about some states being more similar to certain states than to others.[23] In other words, context fixes a relation of comparative similarity among states. Then that relation can be used to say, for any three states, whether or not the first is more similar to the second or to the third (in that context).

For example, consider the following three states: the state of grass being green, the state of grass being turquoise, and the state of grass being red. In the present context, the first state is more similar to the second state than to the third state. The state of grass being green, in this context, is more similar to the state of grass being turquoise than to the state of grass being red. And the comparative similarity relation can be used to capture that.[24]

Now for the truth conditions for counterfactuals. Basically, they say that the truth value of a counterfactual is determined by the parts of sufficiently similar states.

Counterfactuals

Let s be a state in S, and let ϕ and ψ be sentences.

- Verification: s verifies $\phi \mathrel{\Box\!\!\rightarrow} \psi$ if and only if for each state t in S such that t verifies ϕ and s is more similar to t than to any state which does not verify ϕ, there exists a state u in S such that $u \sqsubseteq t$ and u verifies ψ.
- Falsification: s falsifies $\phi \mathrel{\Box\!\!\rightarrow} \psi$ if and only if for some state t in S such that t verifies ϕ and s is more similar to t than to any state which does not verify ϕ, there exists a state u such that $u \sqsubseteq t$ and u falsifies ψ.

In other words, a state s makes a counterfactual true just in case for each state which is sufficiently similar to s and which makes the counterfactual's antecedent true, that state contains a part which makes the counterfactual's consequent true. And a state s makes a counterfactual false just in case for some state which is sufficiently similar to s and which

[23] It is analogous to the comparative similarity relation among possible worlds discussed in (Lewis, 1973; Stalnaker, 1981/1968).

[24] See Appendix F for a more precise characterization of the comparative similarity relation among states.

makes the counterfactual's antecedent true, that state contains a part which makes the counterfactual's consequent false.

For example, consider the sentence '$Wc \mathrel{\Box\!\!\rightarrow} Lc$'. Suppose that '$W$' represents the property of waking up after 9am on July 1 in the year 2021, and suppose that 'c' denotes Charlie. So 'Wc' says that Charlie wakes up after 9am on that particular day. In addition, suppose that 'L' represents the property of being late to a meeting at noon. So 'Lc' says that Charlie is late to the meeting. And therefore, '$Wc \mathrel{\Box\!\!\rightarrow} Lc$' says "If Charlie had woken up after 9am, then Charlie would have been late to the meeting." Let s be a state which describes everything that happens in Charlie's life up to noon on July 1, 2021: so s contains the state of Charlie waking up at 8am on that day; and as a consequence, in s, Charlie is not late. To keep things simple, suppose that there is just one state t such that (i) t verifies 'Wc', and (ii) s is more similar to t than to any state which does not verify 'Wc'. So t is more-or-less exactly like s up to 8am, but regarding events after that time, t and s disagree: in particular, in t, Charlie sleeps past 9am. And let u be the state of Charlie being late to the meeting. Then plausibly, t contains u as a part. So according to Counterfactuals, s verifies '$Wc \mathrel{\Box\!\!\rightarrow} Lc$': that is, s makes the sentence "If Charlie had woken up after 9am, then Charlie would have been late to the meeting" true.

Let us now return to an objection from Section 4. The objection, recall, was this. Given Statespace—and CONTENT in particular—it follows that 'Pa' and '$Pa \vee (Pa \wedge Qb)$' have the same content. But that, along with two assumptions about counterfactuals, has extremely unintuitive implications. The two assumptions are below.

<u>Simplification</u>

For all sentences ϕ, ψ, and χ, $(\phi \vee \chi) \mathrel{\Box\!\!\rightarrow} \psi$ entails $\phi \mathrel{\Box\!\!\rightarrow} \psi$.

<u>Substitution</u>

For all sentences ϕ, ψ, and χ, if ϕ and χ have the same content then $\phi \mathrel{\Box\!\!\rightarrow} \psi$ entails

$\chi \mathbin{\Box\!\!\rightarrow} \psi$.

A simple proof shows that if 'Pa' and '$Pa \vee (Pa \wedge Qb)$' have the same content, then given Simplification and Substitution, the following holds: for all sentences χ, '$Pa \mathbin{\Box\!\!\rightarrow} \chi$' entails '$(Pa \wedge Qb) \mathbin{\Box\!\!\rightarrow} \chi$'. And that, obviously, is a terrible result. As many examples in the literature have shown, strengthening the antecedent of a counterfactual does not always preserve that counterfactual's truth value. The counterfactual "If Charlie had woken up after 9am, then Charlie would have been late to the meeting" may be true, for example, even though the counterfactual "If Charlie had woken up after 9am and the meeting had been moved to the evening, then Charlie would have been late to the meeting" is false.

Now that I have presented the truth conditions for counterfactuals, I can reply to this objection. The reply is simple: Counterfactuals and ENTAILMENT imply that Simplification is false. For consider the counterfactual below.[25]

> (S): "If Spain had fought for either the Allies or the Axis, then Spain would have fought for the Axis."

And consider another counterfactual.

> (S*): "If Spain had fought for the Allies, then Spain would have fought for the Axis."

According to Simplification, (S) entails (S*). But Counterfactuals, along with ENTAILMENT, contradict that. To see why, let s be the state which describes the actual world; note that for historical reasons, s verifies (S). Then take a state t such that (i) t verifies "Spain fought for the Allies," and (ii) s is more similar to t than to any state which does not verify "Spain fought for the Allies." Then plausibly, t does not contain a state which verifies "Spain fought for the

[25] This counterfactual is discussed in (Loewer, 1976; McKay & van Inwagen, 1977).

Axis" as well.[26] So by Counterfactuals, s does not verify (S*). And so by ENTAILMENT, (S) does not entail (S*). Therefore, Simplification is false.

This response, to the objection above, is in good company. Lewis's truth conditions for counterfactuals (1973), and Stalnaker's too (1981/1968), reject Simplification. Of course, there are significant differences between Counterfactuals—and Statespace more generally—and the truth conditions that Lewis and Stalnaker endorse: the latter are based on possible worlds rather than states, for instance. But my approach, based on Statespace, retains one of the core insights of the Lewis/Stalnaker approach: principles like Simplification fail because of the role that comparative similarity plays in the truth conditions for counterfactuals. It is a significant point in favor of Counterfactuals and Statespace, I think, that they retain that core insight.

It is also, in my view, a reason to prefer Counterfactuals over other state-based approaches to counterfactuals in the literature. Some of those other approaches generally endorse Simplification while giving up Substitution (Fine, 2012, p. 232). My approach shows that state-based approaches to counterfactuals can do the reverse: generally endorse Substitution while giving up Simplification.[27] And because I am persuaded by examples like the ones based on (S) and (S*), I prefer state-based accounts of counterfactuals which do that.

8 Logical Subtraction

Statespace can be used to analyze logical subtraction. In this section, I explain how. By way of preparation, I explain what logical subtraction is, and I formulate two conditions which will be important in what follows. Then I present the analysis. Finally, I discuss some

[26]To see why, suppose that t did contain a state like that. Then by (i), t would be a state in which Spain fought for the Axis and the Allies both. But then (ii) would be false, since obviously, s is more similar to plenty of states which only verify "Spain fought for the Allies" than to a state which—like t—verifies both "Spain fought for the Allies" and "Spain fought for the Axis."

[27]I myself am unsure of whether to endorse Substitution, though for lack of space, I will not discuss that here. For more criticisms of state-based approaches to counterfactuals

of the analysis' attractive features.

In broad outline, logical subtraction is the operation of removing one proposition from another. The output of the operation is, intuitively, whatever remains of the second proposition, once the first has been subtracted from it. For example, take the following two propositions: roses are red; and grass is green and roses are red. Subtract the first from the second. The result of that subtraction is, intuitively, the proposition that grass is green. For that is what remains of the second, when the first has been removed from it.

Philosophers often implicitly appeal to logical subtraction. For instance, Wittgenstein asks what remains of the proposition that I raise my arm, once the proposition that my arm goes up has been subtracted from it (1986, p. i6ie). Chalmers describes the mental state of judgment by appealing to subtraction too: the proposition that some mental state is a judgment, Chalmers claims, is what remains after (i) each proposition describing a mental state's phenomenal quality, has been subtracted from (ii) the proposition that the mental state in question is a belief (1996, p. 174). And Goodman's characterization of lawlike sentences can be understood in terms of logical subtraction: the proposition expressed by a lawlike sentence is whatever remains of the proposition expressed by a sentence about a law, once any propositions mentioning truth or falsity have been subtracted from the latter (1955, p. 27).

By way of preparation for the analysis of logical subtraction, I present two conditions which place additional constraints on the complete, well-founded lattice formed by S and $\sqsubseteq$. The first says that certain states can be 'factored out of' greatest lower bounds.

Distributivity

For all subsets A of S, and for all states s in S, $\bigsqcup(A \sqcap s) \sqsubseteq \left(\bigsqcup A\right) \sqcap s$.[28]

Here is the second condition; note that '0' represents the greatest lower bound of all the

[28] $A \sqcap s$ is defined as the set of all states t such that for some a in A, t is $a \sqcap s$.

states in S, and '1' represents the least upper bound of all the states in S.

Existence of Complement

For all states s in S, there exists a state $s^\perp$ in S such that $s \sqcap s^\perp = 1$ and $s \sqcup s^\perp = 0$.

Think of the state $s^\perp$ as the result of subtracting the state s from the state 1. For $s^\perp$ is the state which yields 1, when fused with s. So in a rough but intuitive sense, $s^\perp$ contains everything—from all the states in S—which s lacks.

Here is some more terminology. Say that S and $\sqsubseteq$ jointly form a 'complete, complemented, distributive, well-founded lattice' just in case S and $\sqsubseteq$ jointly form a complete, well-founded lattice which satisfies the two conditions above.

Now for an important result. As I will explain shortly, the theorem below basically says that there is a way to define subtraction among states.[29]

Theorem 3. *Suppose that S and $\sqsubseteq$ form a complete, complemented, distributive, well-founded lattice. Let x and y be states in S such that $x \sqsubseteq y$. Let $Z_{x,y}$ be the set of all states z such that $y = z \sqcap x$. Then $y = \left(\bigsqcup Z_{x,y}\right) \sqcap x$.*

Roughly put, this theorem says the following. Take any states x and y such that y contains x as a part. Then there is a unique smallest state which, when fused to x, yields y. That state is $\bigsqcup Z_{x,y}$, where $Z_{x,y}$ contains all and only the states such that y is the fusion of each such state with x. In other words, intuitively, $\bigsqcup Z_{x,y}$ is the smallest state containing everything in y which x lacks.

Think of theorem 3 as saying that there exists an operation of subtraction among states.[30] More precisely, suppose that S and $\sqsubseteq$ jointly form a complete, complemented,

[29]For the proof, see theorem G.1 in Appendix G.

[30]For a nice discussion of why subtraction might be naturally understood in terms of smallest states—as well as criticisms of an account of logical subtraction due to Hudson (1975)—see (Yablo, 2014, pp. 134–136).

distributive, well-founded lattice. And take any states x and y such that y contains x as a part. Then according to theorem 3, there exists a unique state whose fusion with x is identical to y. It follows that there exists a function which (i) takes in any states x and y such that $x \sqsubseteq y$ as inputs, and (ii) outputs a unique state $\bigsqcup Z_{x,y}$ such that $y = \left(\bigsqcup Z_{x,y}\right) \sqcap x$. In other words, there exists a two-place subtraction operation over all pairs of states for which one state in the pair is part of the other.

It will be helpful to introduce some more notation and terminology. For any states x and y such that $x \sqsubseteq y$, let $(y -' x)$ be the state $\bigsqcup Z_{x,y}$ mentioned in theorem 3. In other words, '$-'$' is the operation of subtraction among states. In what follows, I will often refer to the state $y -' x$ as the 'difference' between y and x.

With that as background, let us analyze the operation of logical subtraction among propositions.[31] Suppose that the proposition $\mathcal{P} = \langle \mathcal{P}_V, \mathcal{P}_F \rangle$ is part of the proposition $\mathcal{Q} = \langle \mathcal{Q}_V, \mathcal{Q}_F \rangle$. Then by Partial Content, a series of parthood relationships obtain among the points of the maximal cones in $\mathcal{P}_V$ and $\mathcal{Q}_V$; and a series of parthood relationships obtain among the points of the maximal cones in $\mathcal{P}_F$ and $\mathcal{Q}_F$ as well. For every point which is a verifier of one proposition and which contains—or is contained by—a point which is a verifier of the other proposition, subtract the contained point from the containing point. Each such state, which results from one of these subtractions, can be used to define a maximal cone in S as a whole. Take the union of all those maximal cones, and let '$\mathcal{Q}_V -^* \mathcal{P}_V$' denote this union. Similarly, for every point which is a falsifier of one proposition and which contains—or is contained by—a point which is a falsifier of the other proposition, subtract the contained point from the containing point. Each such state, which results from one of these subtractions, can be used to define a maximal cone in S as a whole. Take the union of all those maximal cones, and let '$\mathcal{Q}_F -^* \mathcal{P}_F$' denote this union. Then with all that as background, here is the analysis of the operation '$-$' of logical subtraction.

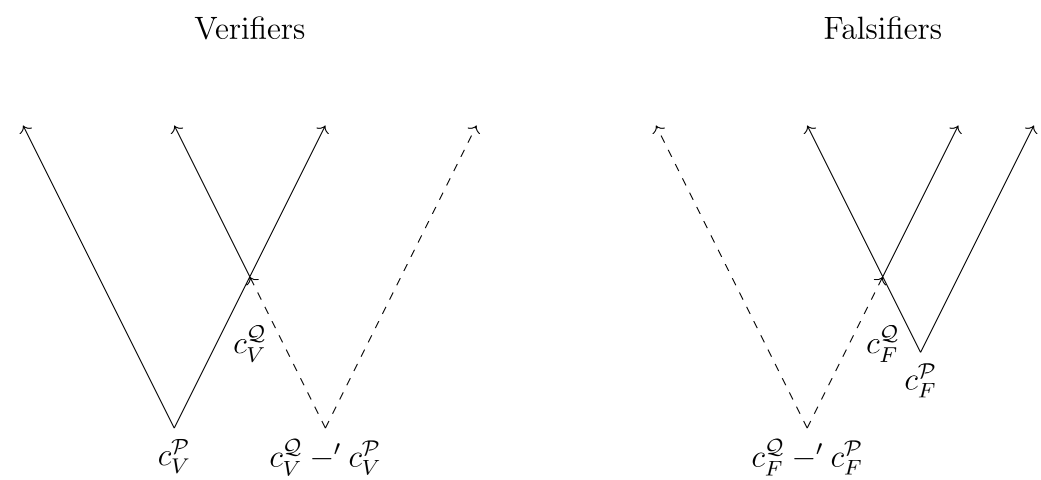
[31] In what follows, I present the basic idea of the analysis; for the fully rigorous analysis, see Appendix G.

Logical Subtraction

Let $\mathcal{P} = \langle \mathcal{P}_V, \mathcal{P}_F \rangle$ and $\mathcal{Q} = \langle \mathcal{Q}_V, \mathcal{Q}_F \rangle$ be propositions such that $\mathcal{P}$ is part of $\mathcal{Q}$. Then $\mathcal{Q} - \mathcal{P} = \langle \mathcal{Q}_V -^* \mathcal{P}_V, \mathcal{Q}_F -^* \mathcal{P}_F \rangle$.

In other words, logical subtraction is the operation of subtracting all (i) contained points in maximal cones of one proposition, from (ii) all containing points in maximal cones of another proposition.

For example, let $\mathcal{P} = \langle C^{\mathcal{P}}_V, C^{\mathcal{P}}_F \rangle$ and $\mathcal{Q} = \langle C^{\mathcal{Q}}_V, C^{\mathcal{Q}}_F \rangle$ be the propositions described by the picture below. The area to the left represents states which are verifiers of $\mathcal{P}$ and of $\mathcal{Q}$, while the area to the right represents states which are falsifiers of $\mathcal{P}$ and of $\mathcal{Q}$.



As the picture shows—and for the sorts of reasons discussed in Section 4—$\mathcal{P}$ is part of $\mathcal{Q}$. The difference between $c^{\mathcal{Q}}_V$ and $c^{\mathcal{P}}_V$ is represented by the state $c^{\mathcal{Q}}_V -' c^{\mathcal{P}}_V$, and the difference between $c^{\mathcal{Q}}_F$ and $c^{\mathcal{P}}_F$ is represented by the state $c^{\mathcal{Q}}_F -' c^{\mathcal{P}}_F$. As in previous pictures, for any two solid lines emanating from a single state, the area between those lines—along with the lines themselves—represents the maximal cone whose point is that state. And as in previous pictures, if there is a line—solid or dashed—from one state to another, then the latter contains the former as a part. So both $c^{\mathcal{P}}_V$ and $c^{\mathcal{Q}}_V -' c^{\mathcal{P}}_V$ are parts of $c^{\mathcal{Q}}_V$, and both $c^{\mathcal{P}}_F$ and $c^{\mathcal{Q}}_F -' c^{\mathcal{P}}_F$ are parts of $c^{\mathcal{Q}}_F$. The area between the two dashed lines on the left—including the portion of the area which overlaps the cone whose point is $c^{\mathcal{Q}}_V$, where the left-most dashed line becomes solid—represents the maximal cone whose point is $c_V^{\mathcal{Q}} -' c_V^{\mathcal{P}}$. That cone is the set of verifiers $C_{c_V^{\mathcal{Q}} -' c_V^{\mathcal{P}}} = C_V^{\mathcal{Q}} -^* C_V^{\mathcal{P}}$. Analogously, the area between the two dashed lines on the right represents the maximal cone whose point is $c_F^{\mathcal{Q}} -' c_F^{\mathcal{P}}$. That cone is the set of verifiers $C_{c_F^{\mathcal{Q}} -' c_F^{\mathcal{P}}} = C_F^{\mathcal{Q}} -^* C_F^{\mathcal{P}}$. So together, these two cones—the ones delimited by the dashed lines—represent the proposition $\mathcal{Q} - \mathcal{P}$.

For lack of space, I will not discuss Logical Subtraction further. But very briefly, it is worth pointing out the following virtues of it. First, Logical Subtraction can be used to define various other subtraction relations which have intuitive implications. For instance, Logical Subtraction can be used to define a subtraction operation that yields intuitively correct results for conjunction: this subtraction operation implies that in general, and in a certain precise sense, the positive content of a conjunct subtracted from the positive content of a conjunction is the positive content of the other conjunct. In addition, Logical Subtraction can be used to define a subtraction operation that yields intuitively correct results for disjunctions: this subtraction operation implies that in general, and in a certain precise sense, the positive content of a disjunct subtracted from the positive content of a disjunction is the positive content of the other disjunct.

9 Comparisons

In this section, I compare Statespace to other theories of states and truth—call them 'orthodox theories'—in the literature. As will become clear, despite some similarities, Statespace is quite different from orthodox theories. And as I will argue, those differences favor Statespace.

Before continuing, however, it is worth making a quick disclaimer. Though my discussion of orthodox theories will be critical, I am very sympathetic to the guiding idea which underwrites orthodox theories. According to that guiding idea, truth conditions for first-order languages should be formulated in terms of states, verification, and falsification; possible worlds, in other words, are often not good enough (Fine, 2017a; 2017b; 2017c; Kratzer 1989; 2012). Statespace is a specific implementation of that guiding idea, just as many orthodox theories are. So ultimately, I think that orthodox theories—in addition to being beautiful and elegant approaches to semantics in their own right—are hiking in the right direction. This section is merely for quibbles over the specifics of the route.

Orthodox theories[32] posit a set of states S, and a partial order $\sqsubseteq$ over S, such that S and $\sqsubseteq$ jointly satisfy some of the conditions from Section 2. In addition, orthodox theories use states to formulate verification conditions and falsification conditions, some of which are equivalent to the conditions from Section 3.[33]

Nevertheless, there are at least three significant reasons to prefer Statespace over orthodox theories. The reasons stem from the fact that Statespace endorses more natural conditions for conjunctive formulas, disjunctive formulas, and quantificational formulas. Orthodox theories place more structural restrictions on what it takes for a state to verify, or falsify, formulas like those. And as will become clear, those restrictions are unideal.

The first reason concerns quantificational formulas. According to orthodox theories, whether a state verifies a universal or an existential depends—in large part—on exactly which constants are in the relevant first-order language. In particular, suppose that the first-order language contains fewer constants than objects. Then according to the verification conditions adopted by orthodox theories, a state may verify a sentence of the form "Everything is thus-and-so" even if, intuitively, it should not. This happens when there are no constants, in the relevant first-order language, which name the things that are not thus-and-so. Analogous problems arise for the other truth conditions, for quantifiers, which orthodox theories endorse.

Statespace faces no such problems. For in Statespace, the verification and falsification

[32]I focus on the theories discussed by Elgin (2021), Fine (2017a; 2017b; 2017c), and others. For lack of space, I cannot discuss other analogous systems here, such as those formulated by Angell (1977) and Correia (2016). My criticisms can be adapted into criticisms of those systems too, however.

[33]For lack of space, I will not discuss the theory of truth and content proposed by Yablo (2014). Yablo's theory is based on sets of possible worlds, rather than states. For discussion of why states may be better than sets of possible worlds, when formulating theories of content, see (Fine, 2020); for replies, see (Yablo, 2018).

conditions do not mention constants at all. Those conditions only mention (i) objects, and (ii) ways of interpreting variables. So according to Statespace, whether a state verifies a universal or an existential does not depend on arbitrary features of the first-order language in question: it does not matter, for instance, if the first-order language contains fewer constants than objects.[34] And that is a reason to prefer Statespace over orthodox theories.

The second reason, for preferring Statespace, concerns important structural similarities—that is, important dualities—among the different logical symbols of first-order languages. For simplicity, let us focus on the duality between conjunction and disjunction.[35] According to Statespace, a state verifies a conjunction just in case that state contains two parts, one of which verifies one conjunct and one of which verifies the other conjunct. And according to Statespace, a state verifies a disjunction just in case that state contains a part which verifies either one disjunct or the other. In other words, according to Statespace, verifying a conjunction is a matter of having parts which verify both conjuncts, and verifying a disjunction is a matter of having parts which verify at least one disjunct. So Statespace does a good job of respecting the truth-functional duality between conjunction and disjunction: in both cases, verification is a matter of having parts which verify certain dualing combinations of conjuncts and/or disjuncts.[36]

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[34]Put more technically, Statespace avoids this problem because it uses variable assignments; see Appendix B.

[35]In what follows, I discuss duality informally, as a kind of structural symmetry. For a fully rigorous theory of duality—in logic, and in other areas too—see (Awodey, 2010).

[36]Here is another way of thinking about this duality: as a simple but tedious proof shows, there is an elegant correspondence between (i) the verification and falsification conditions for negation, conjunction, and disjunction in Statespace, and (ii) the truth conditions for negation, conjunction, and disjunction in the four-valued logic of Belnap (2019) and Dunn (2019). In particular, the correspondence works like this. Say that a state stands in relation N to a sentence just in case that state neither verifies nor falsifies that sentence. Say that a state stands in relation T to a sentence just in case that state verifies, but does not falsify, that sentence. Say that a state stands in relation F to a sentence just in case that state falsifies, but does not verify, that sentence. And say that a state stands in relation B to a sentence just in case that state both verifies and falsifies that sentence. Call these the 'truth-relevance' relations. Then the following holds: for all states s and all sentences ϕ and ψ,

(i) if s stands in a given truth-relevance relation R to ϕ then s stands in the truth-relevance relation R' to $\neg\phi$ given by the Belnap-Dunn truth table for negation,

(ii) if s stands in a given truth-relevance relation R to ϕ and s stands in a given truth-relevance relation R' to ψ then s stands in the truth-relevance relation R'' to $\phi \wedge \psi$ given by the Belnap-Dunn truth table for conjunction, and

(iii) if s stands in a given truth-relevance relation R to ϕ and s stands in a given truth-relevance relation

Orthodox theories, however, do not respect the truth-functional duality between conjunctions and disjunctions. For consider the following two verification conditions which orthodox theories endorse.

Conjunction$_O$

A state verifies a conjunction just in case *that state is the least upper bound* of two states, one of which verifies one conjunct and one of which verifies the other conjunct.

Disjunction$_O$

A state verifies a disjunction just in case *that state itself* verifies at least one of the disjuncts.

These verification conditions are not duals of each other: they are not, that is, structurally similar to one another, in the ways that truth conditions for conjunction and disjunction should be. To see why, it helps to consider the following, alternative accounts of verification for conjunction and disjunction.

Conjunction^2_O

A state verifies a conjunction just in case *that state itself* verifies both of the conjuncts.

Disjunction^2_O

A state verifies a disjunction just in case *that state is the greatest lower bound* of two states, one of which verifies one disjunct and one of which verifies the other disjunct.

Conjunction$_O$ is more structurally similar to Disjunction^2_O than to Disjunction$_O$: for whereas

R' to ψ then s stands in the truth-relevance relation R'' to $\phi \vee \psi$ given by the Belnap-Dunn truth table for disjunction.

Conjunction_O and Disjunction^2_O both take verifiers to be certain sorts of bounds, Disjunction_O does not. Likewise, Disjunction_O is more structurally similar to Conjunction^2_O than to Conjunction_O: for whereas Disjunction_O and Conjunction^2_O both take the verifiers of a more complex sentence to be verifiers of the parts of that sentence, Conjunction_O does not. So a theory which endorses Conjunction_O should endorse Disjunction^2_O; and a theory which endorses Disjunction_O should endorse Conjunction^2_O. But that is not what orthodox theories, of verification and falsification, actually do. Orthodox theories endorse Conjunction_O and Disjunction_O. And that is an unattractive, unideal combination of conditions to endorse. That endorsement breaks the structural similarity—the duality—between conjunction and disjunction.[37]

An entirely analogous problem arises for many other conditions which orthodox theories endorse. For instance, orthodox theories endorse structurally dissimilar falsification conditions for conjunctions and disjunctions. And orthodox theories endorse structurally dissimilar verification conditions, and falsification conditions too, for universals and existentials.

Statespace does not. When it comes to verification and falsification conditions for conjunction and disjunction—and for the quantifiers too—Statespace does a better job of respecting important structural similarities. Statespace captures the dualisms which that logical vocabulary exemplifies. And that is a reason to prefer Statespace.[38]

The third reason for preferring Statespace is, in a sense, more basic that the previous two. Roughly put, Statespace does a better job of adhering to the intuitive, guiding, underlying idea of state-based approaches to truth conditions. According to that idea, a given state makes a given sentence true, or false, in virtue of having parts which make the subformulas of that sentence true or false. To make a sentence true is just to have parts which make the

[37] The reason for this, which I cannot get into here, is that given other postulates of orthodox theories, endorsing both Conjunction_O and Disjunction^2_O—or endorsing both Conjunction^2_O and Disjunction_O—would have extremely problematic consequences.

[38] That is also a reason for preferring Statespace over a theory of verification and falsification due to van Fraassen (1969). Van Fraassen's theory, a kind of precursor to the contemporary orthodox theories on which I have focused here, also gives structurally dissimilar truth conditions for '$\wedge$' and '$\vee$', and for '$\forall$' and '$\exists$' too (1969, p. 484-486).

parts of that sentence true. And to make a sentence false is just to have parts which make the parts of that sentence false. That—according to the intuitive, guiding, underlying idea of state-based approaches—is what truth and falsity is.

Statespace respects this. To see why, just look at the conditions Conjunction, Disjunction, Universal, and Existential. Each of those conditions has the following basic form: a state makes a sentence true just in case, roughly, that state has parts which make the parts of that sentence true. So according to Statespace, to make a sentence true or false is nothing more—and nothing less—than having parts which make the parts of that sentence true or false, respectively. So Statespace does a good job of adhering to the intuitive, guiding, underlying idea of state-based approaches to truth conditions.

Not so, however, for orthodox theories. For according to orthodox theories, there is *more* to making a sentence true than having parts which make the parts of the sentence true. And there is *more* to making a sentence false than having parts which make the parts of the sentence false. For instance, take conjunctions. Recall that orthodox theories endorse Conjunction$_O$: a state makes a conjunction true just in case, roughly, (i) that state has parts which make the two conjuncts true, but also (ii) those parts' least upper bound is that state itself. So orthodox theories adopt an additional clause in their formulation of the verification condition for conjunction. Orthodox theories do similarly for their other verification conditions, and falsification conditions, too. And that is a reason to prefer Statespace over orthodox theories.

So Statespace represents a significant improvement over orthodox theories of how states determine the truth values of sentences. Whereas orthodox theories imply that the verification and falsification conditions for quantificational sentences depend on arbitrary features of the language in question, Statespace does not. Whereas orthodox theories do not respect certain important similarities between certain logical symbols of first-order languages, Statespace does. And Statespace does a better job of adhering to the intuitive, guiding, underlying idea of state-based approaches to truth conditions.

10 Conclusion

Statespace supports an extremely rich theory of meaning. According to Statespace, states form a complete, well-founded lattice, ordered by a parthood relation. Moreover, according to Statespace, sentences in first-order logic are made true and made false by states in complete, well-ordered lattices: six conditions describe these relations of verification and falsification.

As has been shown, Statespace can be used to analyze many notions related to sentences and their meanings. Here is a summary of what I analyzed in this paper:

- the contents of sentences,
- propositions,
- positive partial content,
- negative partial content,
- partial content,
- exact verification,
- exact falsification,
- subject matters,
- aboutness,
- entailment,
- containment,
- counterfactual truth conditions, and
- logical subtraction.

Each analysis, of the above, is independently plausible, relatively simple, philosophically significant, and formally rigorous.

So there is much to like about Statespace. It is an attractive theory of states, sentences, verification, and falsification. It supports attractive analyses of many different notions related to sentences and their meanings. And it has some advantages over alternative accounts of verification and falsification. Altogether, Statespace is worth taking seriously.

A The Mereology of States

Let S be a set of states. Then a two-place relation $\sqsubseteq$ over S is a 'partial order' just in case the following holds.

1. Reflexivity: for all $s \in S$, $s \sqsubseteq s$.
2. Anti-symmetry: for all $s, t \in S$, if $s \sqsubseteq t$ and $t \sqsubseteq s$ then $s = t$.
3. Transitivity: for all $r, s, t \in S$, if $r \sqsubseteq s$ and $s \sqsubseteq t$ then $r \sqsubseteq t$.

For each $A \subseteq S$ and each $s \in S$, s is an 'upper bound' of A just in case for all $a \in A$, $a \sqsubseteq s$. For each $A \subseteq S$ and each $s \in S$, s is a 'lower bound' of A just in case for all $a \in A$, $s \sqsubseteq a$. In addition, for each $A \subseteq S$ and each $s \in S$, s is a 'least upper bound' of A just in case (i) s is an upper bound of A, and (ii) for all upper bounds t of A, $s \sqsubseteq t$. And for each $A \subseteq S$ and each $s \in S$, s is a 'greatest lower bound' of A if and only if (i) s is a lower bound of A, and (ii) for all lower bounds t of A, $t \sqsubseteq s$. The pair $(S, \sqsubseteq)$ is a 'complete lattice' just in case S is a set of states, $\sqsubseteq$ is a partial order over S, and for all $A \subseteq S$, the least upper bound $\bigsqcap A$ exists and is in S, and the greatest lower bound $\bigsqcup A$ exists and is in S. And the complete lattice $(S, \sqsubseteq)$ is 'well-founded' just in case there is no infinite sequence $s_1, s_2, \ldots \in S$ such that $s_1 \sqsupsetneq s_2 \sqsupsetneq \cdots$.

Definitions concerning cones will be relevant in later appendices. Let $(S, \sqsubseteq)$ be a complete lattice. Say that $C \subseteq S$ is a 'cone' just in case for some $c \in S$, $C = \{s \in S \mid c \sqsubseteq s\}$. In addition, for each $A \subseteq S$, say that C is a 'maximal cone in A' just in case (i) C is a cone in S, (ii) $C \subseteq A$, and (iii) for all $C' \subseteq A$ such that C' is a cone in S and $C \subseteq C'$, $C = C'$.

Definitions concerning chains will be relevant in later appendices too. Let $(S, \sqsubseteq)$ be a complete lattice. Say that $C \subseteq S$ is a 'chain' just in case for all $x, y \in C$, either $x \sqsubseteq y$ or $y \sqsubseteq x$. In addition, for each $A \subseteq S$, say that $C \subseteq S$ is a 'maximal chain in A' just in case (i) C is a chain, (ii) $C \subseteq A$, and (iii) for all $C' \subseteq A$ such that C' is a chain and $C \subseteq C'$, $C = C'$.

The following lemma says that every chain is contained in a maximal chain.

Lemma A.1. *Let S be a set and let $\sqsubseteq$ be a partial order over S. Then for each $B \subseteq S$ and for each chain C such that $C \subseteq B$, there exists a maximal chain C' in B which contains C.*

For a proof, see (Frink, 1952).

One other definition is needed. Let $(S, \sqsubseteq)$ be a complete lattice. Say that $A \subseteq S$ is 'upward closed' just in case for all $a \in A$ and all $s \in S$ such that $a \sqsubseteq s$, $s \in A$.

B Truth Conditions

In this appendix, I present the fully rigorous truth conditions for sentences of L. To start, I define several different sorts of functions. Then I define the models. Finally, I present the truth conditions.

The following two types of functions will be important in what follows. First, a 'constant assignment' is a function which maps each constant in L to an object. Second, a 'variable assignment' is a function which maps each variable in L to an object.

The following piece of notation will also be helpful. Let χ be a variable, let ϕ be a formula in which χ appears free, and let κ be a constant. Then $\phi[\kappa/\chi]$ is the formula which results from replacing each free occurrence of χ in ϕ with κ.

A 'model' is a six-tuple $(S, \sqsubseteq, I, j, |\cdot|^+_\cdot, |\cdot|^-_\cdot)$ with the following features. First, S is a set of states, and $\sqsubseteq$ is a two-place parthood relation over S such that $(S, \sqsubseteq)$ is a well-founded, complete lattice. I is a set of objects, and j is a constant assignment which maps each constant in L to an object in I. Finally, the two-place functions $|\cdot|^+_\cdot$ and $|\cdot|^-_\cdot$—which map atomic formulas and variable assignments to sets of states in S—obey the following constraints.[39]

[39]The functions $|\cdot|^+_\cdot$ and $|\cdot|^-_\cdot$ are the precisified versions of the functions $V_\_$ and $F_\_$ that feature in the first truth condition in Section 3.

1. For each atomic formula $\mathcal{F}\tau_1 \ldots \tau_n$ and for each variable assignment g, $|\mathcal{F}\tau_1 \ldots \tau_n|_g^+$ is a cone and $|\mathcal{F}\tau_1 \ldots \tau_n|_g^-$ is a cone.
2. For each atomic formula $\mathcal{F}\tau_1 \ldots \tau_n$ and for all variable assignments g and g', if g and g' agree on which variables—among the terms $\tau_1, \ldots, \tau_n$—are mapped to which objects in I, then $|\mathcal{F}\tau_1 \ldots \tau_n|_g^+ = |\mathcal{F}\tau_1 \ldots \tau_n|_{g'}^+$ and $|\mathcal{F}\tau_1 \ldots \tau_n|_g^- = |\mathcal{F}\tau_1 \ldots \tau_n|_{g'}^-$.
3. For each variable χ, for each atomic formula ϕ in which χ appears free, for each constant κ, and for each variable assignment g, if $g(\chi) = j(\kappa)$ then $|\phi|_g^+ = |\phi[\kappa/\chi]|_g^+$, and $|\phi|_g^- = |\phi[\kappa/\chi]|_g^-$.

The second constraint says that interpretations of atomic formulas cannot differ, if those interpretations agree on what the variables in those formulas denote. The third constraint says that if a variable is taken to denote a specific object, and if some constant also denotes that object, then interpretations of atomic formulas which feature that variable cannot differ from interpretations of atomic formulas in which that variable has been replaced by that constant.

Now for the truth conditions. Let M be a model $(S, \sqsubseteq, I, j, |\cdot|_\cdot^+, |\cdot|_\cdot^-)$ whose elements are as described above. Then truth in L—relative to M—is defined recursively; note that in what follows, '$\Vdash_g$' is the relation of verification (relative to variable assignment g), and '$\dashv\vert_g$' is the relation of falsification (relative to variable assignment g).

(1) For all $s \in S$, variable assignments g, natural numbers n, n-place predicates $\mathcal{F}$, and terms $\tau_1, \ldots, \tau_n$, $s \Vdash_g \mathcal{F}\tau_1 \ldots \tau_n$ if and only if $s \in |\mathcal{F}\tau_1 \ldots \tau_n|_g^+$.

(2) For all $s \in S$, variable assignments g, natural numbers n, n-place predicates $\mathcal{F}$, and terms $\tau_1, \ldots, \tau_n$, $s \dashv\vert_g \mathcal{F}\tau_1 \ldots \tau_n$ if and only if $s \in |\mathcal{F}\tau_1 \ldots \tau_n|_g^-$.

(3) For all $s \in S$, variable assignments g, and formulas ϕ, $s \Vdash_g \neg\phi$ if and only if $s \dashv\vert_g \phi$.

(4) For all $s \in S$, variable assignments g, and formulas ϕ, $s \dashv\vert_g \neg\phi$ if and only if $s \Vdash_g \phi$.

(5) For all $s \in S$, variable assignments g, and formulas ϕ and ψ, $s \Vdash_g \phi \wedge \psi$ if and only if there are $t \sqsubseteq s$ and $u \sqsubseteq s$ such that $t \Vdash_g \phi$ and $u \Vdash_g \psi$.

(6) For all $s \in S$, variable assignments g, and formulas ϕ and ψ, $s \dashv\vert_g \phi \wedge \psi$ if and only if

for some $t \sqsubseteq s$, either $t \dashv\!|_g \phi$ or $t \dashv\!|_g \psi$.

(7) For all $s \in S$, variable assignments g, and formulas ϕ and ψ, $s \Vdash_g \phi \vee \psi$ if and only if for some $t \sqsubseteq s$, either $t \Vdash_g \phi$ or $t \Vdash_g \psi$.

(8) For all $s \in S$, variable assignments g, and formulas ϕ and ψ, $s \dashv\!|_g \phi \vee \psi$ if and only if there are $t \sqsubseteq s$ and $u \sqsubseteq s$ such that $t \dashv\!|_g \phi$ and $u \dashv\!|_g \psi$.

(9) For all $s \in S$, variable assignments g, variables χ, and formulas $\phi(\chi)$ in which χ appears free, $s \Vdash_g \forall\chi\phi(\chi)$ if and only if for all variable assignments g' which differ from g at most on χ, there is an $s' \sqsubseteq s$ such that $s' \Vdash_{g'} \phi(\chi)$.

(10) For all $s \in S$, variable assignments g, variables χ, and formulas $\phi(\chi)$ in which χ appears free, $s \dashv\!|_g \forall\chi\phi(\chi)$ if and only if for some variable assignment g' which differs from g at most on χ, there is an $s' \sqsubseteq s$ such that $s' \dashv\!|_{g'} \phi(\chi)$.

(11) For all $s \in S$, variable assignments g, variables χ, and formulas $\phi(\chi)$ in which χ appears free, $s \Vdash_g \exists\chi\phi(\chi)$ if and only if for some variable assignment g' which differs from g at most on χ, there is an $s' \sqsubseteq s$ such that $s' \Vdash_{g'} \phi(\chi)$.

(12) For all $s \in S$, variable assignments g, variables χ, and formulas $\phi(\chi)$ in which χ appears free, $s \dashv\!|_g \exists\chi\phi(\chi)$ if and only if for all variable assignments g' which differ from g at most on χ, there is an $s' \sqsubseteq s$ such that $s' \dashv\!|_{g'} \phi(\chi)$.

Here are some more definitions. For all formulas ϕ and variable assignments g, let $V_{\phi,g} = \{s \in S \mid s \Vdash_g \phi\}$ and let $F_{\phi,g} = \{s \in S \mid s \dashv\!|_g \phi\}$. So $V_{\phi,g}$ is the set of all states which verify ϕ (relative to g), and $F_{\phi,g}$ is the set of all states which falsify ϕ (relative to g).

C Contents, Propositions, and Parts

In this appendix, I prove the theorem—mentioned in Section 4—which says that verifiers and falsifiers are unions of maximal cones. By way of preparation, I prove a series of additional theorems and lemmas.

Lemma C.1. *For each formula ϕ of L and for each variable assignment g, $V_{\phi,g}$ is non-empty and upward closed, and $F_{\phi,g}$ is non-empty and upward closed.*

Proof. The proof is by induction on the complexity of formulas. The inductive hypothesis is this: for each formula ϕ of complexity less than n, and for each variable assignment g, $V_{\phi,g}$ is non-empty and upward closed, and $F_{\phi,g}$ is non-empty and upward closed.

For the base case, let ϕ be an atomic formula. For all variable assignments g, $|\phi|_g^+$ and $|\phi|_g^-$ are cones; so they are non-empty and upward closed. So by (1), for each variable assignment g, $V_{\phi,g}$ is non-empty and upward closed. And by (2), for each variable assignment g, $F_{\phi,g}$ is non-empty and upward closed too. This establishes the base case.

For the inductive step, there are five cases to consider. First, for some formula ψ, ϕ is $\neg\psi$. Second, for some formulas ϕ_1 and ϕ_2, ϕ is $\phi_1 \wedge \phi_2$. Third, for some formulas ϕ_1 and ϕ_2, ϕ is $\phi_1 \vee \phi_2$. Fourth, for some formula ψ and some variable χ, ϕ is $\forall\chi\psi(\chi)$. Fifth, for some formula ψ and some variable χ, ϕ is $\exists\chi\psi(\chi)$. Let us consider each of these in turn.

First, suppose that ϕ is $\neg\psi$. By the inductive hypothesis, for all variable assignments g, $F_{\psi,g}$ is non-empty and upward closed, and $V_{\psi,g}$ is non-empty and upward closed. So (3) and (4) imply that for all variable assignments g, $V_{\neg\psi,g}$ is non-empty and upward closed, and $F_{\neg\psi,g}$ is non-empty and upward closed.

Second, suppose that ϕ is $\phi_1 \wedge \phi_2$. The inductive hypothesis implies that for all variable assignments g, $V_{\phi_1,g}$, $F_{\phi_1,g}$, $V_{\phi_2,g}$, and $F_{\phi_2,g}$ are all non-empty and upward closed. To show that $V_{\phi,g}$ and $F_{\phi,g}$ are non-empty, take $s_1 \in V_{\phi_1,g}$, $t_1 \in F_{\phi_1,g}$, $s_2 \in V_{\phi_2,g}$, and $t_2 \in F_{\phi_2,g}$. Let $s = s_1 \sqcap s_2$, and let $t = t_1 \sqcap t_2$. Since there are states s_1 and s_2 such that $s_1 \sqsubseteq s$, $s_2 \sqsubseteq s$, $s_1 \Vdash_g \phi_1$, and $s_2 \Vdash_g \phi_2$, (5) implies that $s \Vdash_g \phi_1 \wedge \phi_2$; so $V_{\phi,g}$ is non-empty. An analogous argument, along with (6), shows that $t \dashv\vert_g \phi_1 \wedge \phi_2$; so $F_{\phi,g}$ is non-empty.

Now let us show that $V_{\phi,g}$ and $F_{\phi,g}$ are upward closed. Suppose that $r \in V_{\phi,g}$, $s \in S$, and $r \sqsubseteq s$. Along with the fact that $r \sqsubseteq s$, (5) implies that there are states r_1 and r_2 such that $r_1 \sqsubseteq s$, $r_2 \sqsubseteq s$, $r_1 \Vdash_g \phi_1$, and $r_2 \Vdash_g \phi_2$. So by (5) again, $s \in V_{\phi,g}$; and so $V_{\phi,g}$ is upward closed. A similar line of reasoning shows that $F_{\phi,g}$ is upward closed too.

Third, suppose that ϕ is $\phi_1 \vee \phi_2$. As shown by an argument entirely analogous to the one for the case where ϕ was $\phi_1 \wedge \phi_2$, for each variable assignment g, $V_{\phi,g}$ and $F_{\phi,g}$ are both non-empty. The proof that $V_{\phi,g}$ is upward closed is entirely analogous to the proof that $F_{\phi_1 \wedge \phi_2,g}$ is upward closed. And the proof that $F_{\phi,g}$ is upward closed is entirely analogous to the proof that $V_{\phi_1 \wedge \phi_2,g}$ is upward closed.

Fourth, suppose that ϕ is $\forall\chi\psi(\chi)$. Let g be a variable assignment. By the inductive hypothesis, for each variable assignment g' which differs from g at most on χ, $V_{\psi(\chi),g'}$ is non-empty and upward closed, and $F_{\psi(\chi),g'}$ is non-empty and upward closed. To start, let us show that $V_{\phi,g}$ and $F_{\phi,g}$ are non-empty. Towards that end, note that for each such g', there are states $s_{g'}$ and $t_{g'}$ such that $s_{g'} \Vdash_{g'} \psi(\chi)$ and $t_{g'} \dashv\mid_{g'} \psi(\chi)$. Let S' be the set of all these $s_{g'}$, and let T' be the set of all these $t_{g'}$. Let $s = \bigsqcap S'$ and let $t = \bigsqcap T'$. For each variable assignment g' which differs from g at most on χ, there is a state $s' \sqsubseteq s$—namely, $s' = s_{g'}$—such that $s' \Vdash_{g'} \psi(\chi)$. So by (9), $s \Vdash_g \forall\chi\psi(\chi)$. Therefore, $V_{\phi,g}$ is non-empty. An entirely analogous line of reasoning shows that $F_{\phi,g}$ is non-empty too.

Now let us show that $V_{\phi,g}$ and $F_{\phi,g}$ are upward closed. Suppose that $r \in V_{\phi,g}$, $s \in S$, and $r \sqsubseteq s$. Along with the fact that $r \sqsubseteq s$, (9) implies that for each g' which differs from g at most on χ, there is a state $r' \sqsubseteq s$ such that $r' \Vdash_{g'} \psi(\chi)$. So by (9) again, $s \in V_{\phi,g}$; in other words, $V_{\phi,g}$ is upward closed. A similar line of reasoning shows that $F_{\phi,g}$ is upward closed too.

Fifth, suppose that ϕ is $\exists\chi\psi(\chi)$. As shown by an argument entirely analogous to the one for the case where ϕ was $\forall\chi\psi(\chi)$, for each variable assignment g, $V_{\phi,g}$ and $F_{\phi,g}$ are both non-empty. The proof that $V_{\phi,g}$ is upward closed is entirely analogous to the proof that $F_{\forall\chi\psi(\chi),g}$ is upward closed. And the proof that $F_{\phi,g}$ is upward closed is entirely analogous to the proof that $V_{\forall\chi\psi(\chi),g}$ is upward closed. □

Here are some more useful definitions. Let G be the set of all variable assignments. For all formulas ϕ, let $V_\phi = \bigcap_{g \in G} V_{\phi,g}$, and let $F_\phi = \bigcap_{g \in G} F_{\phi,g}$.

The following theorem says that the verifiers and falsifiers of sentences—which are verifiers and falsifiers of those sentences for each and every variable assignment whatsoever—both (i) exist, and (ii) form upward closed sets.

Theorem C.1. *For each sentence ϕ in L, V_ϕ is non-empty and upward closed, and F_ϕ is non-empty and upward closed.*

Proof. Let ϕ be a sentence in L. To start, let us show that both V_ϕ and F_ϕ are non-empty. By lemma C.1, for each variable assignment g, there exists $s_g \in V_{\phi,g}$ and $t_g \in F_{\phi,g}$. Let S be the set of all these s_g, and let T be the set of all these t_g. Let $s = \bigsqcap S$, and let $t = \bigsqcap T$. Note that for all g, $s_g \sqsubseteq s$ and $t_g \sqsubseteq t$. Therefore, since lemma C.1 implies that $V_{\phi,g}$ and $F_{\phi,g}$ are upward closed for each g, it follows that for each g, $s \in V_{\phi,g}$ and $t \in F_{\phi,g}$. Therefore, $s \in V_\phi$ and $t \in F_\phi$; so V_ϕ and F_ϕ are non-empty.

Now let us show that both V_ϕ and F_ϕ are upward closed. Towards that end, suppose that $r \in V_\phi$, $s \in S$, and $r \sqsubseteq s$. Then for each variable assignment g, $r \in V_{\phi,g}$. Since lemma C.1 implies that each such $V_{\phi,g}$ is upward closed, it follows that for each g, $s \in V_{\phi,g}$. Therefore, $s \in V_\phi$; so V_ϕ is upward closed. An entirely analogous argument shows that F_ϕ is upward closed too. □

The following lemma says that every maximal chain of verifiers, and every maximal chain of falsifiers, contains its greatest lower bound.

Lemma C.2. *Let ϕ be a sentence. If C is a chain in V_ϕ then $\bigsqcup C \in C$, and if C is a chain in F_ϕ then $\bigsqcup C \in C$.*

Proof. Let C be a chain in V_ϕ. Since $\sqsubseteq$ is well-founded, it is not the case that there exist states $s_1, s_2, \ldots \in C$ such that $s_1 \sqsupsetneq s_2 \sqsupsetneq \cdots$. Along with the fact that C is a chain, it follows that there exists a state $c \in C$ such that for all $s \in C$, $c \sqsubseteq s$. The definition of greatest lower bound, along with the fact that $c \in C$, implies that $c = \bigsqcup C$. Therefore, $\bigsqcup C \in C$. An entirely analogous argument shows that if C is a chain in F_ϕ, then $\bigsqcup C \in C$. □

The following theorem—mentioned in Section 4—describes the general structure of verifiers and falsifiers. It says that each sentence's verifiers, and falsifiers, are basically just unions of maximal cones.

Theorem C.2 (theorem 1 in the main text)**.** *Let ϕ be a sentence. Then V_ϕ is the union of cones which are maximal in V_ϕ, and F_ϕ is the union of cones which are maximal in F_ϕ.*

Proof. For each $v \in V_\phi$, lemma A.1 implies that there is a chain $C_v \subseteq S$ which is maximal in V_ϕ and which contains v. For each such v, let $c_v = \bigsqcup C_v$. By lemma C.2, for each such v, $c_v \in C_v$. By theorem C.1, and in particular the fact that V_ϕ is upward closed, the cone $A_v = \{s \in S \mid c_v \sqsubseteq s\}$ is a subset of V_ϕ. In addition, note that since C_v is maximal, then for all $w \in V_\phi$ such that $w \sqsubseteq c_v$, $w = c_v$. Therefore, the cone A_v is maximal in V_ϕ. Let $\mathcal{M}$ be the set of all these cones A_v. Then $\bigcup \mathcal{M} = V_\phi$. So V_ϕ is the union of cones which are maximal in V_ϕ. An entirely analogous argument shows that F_ϕ is the union of cones which are maximal in F_ϕ. □

D Exact Verification, Exact Falsification, Subject Matter, and Aboutness

Here is the fully rigorous definition of subject matter. Let ϕ be a sentence. By theorem C.2, there exists a set of cones $\mathcal{C}_V$—each of which is maximal in V_ϕ—such that $V_\phi = \bigcup \mathcal{C}_V$. For each $C \in \mathcal{C}_V$, let v_C be the point of C. Then let $v_\phi = \bigsqcap_{C \in \mathcal{C}_V} v_C$. Similarly, by theorem C.2, there exists a set of cones $\mathcal{C}_F$—each of which is maximal in F_ϕ—such that $F_\phi = \bigcup \mathcal{C}_F$. For each $C \in \mathcal{C}_F$, let f_C be the point of C. Then let $f_\phi = \bigsqcap_{C \in \mathcal{C}_F} f_C$. Finally, the subject matter of ϕ is $\langle v_\phi, f_\phi \rangle$.

The following theorem—mentioned in Section 5—says that the positive subject matter of a sentence verifies that sentence, and the negative subject matter of a sentence falsifies that sentence.

Theorem D.1 (theorem 2 in the main text)**.** *Let ϕ be a sentence with subject matter $\langle v_\phi, f_\phi \rangle$. Then $v_\phi \in V_\phi$ and $f_\phi \in F_\phi$.*

Proof. Let $\mathcal{C}^V$ and $\mathcal{C}^F$ be as defined above. For each $C \in \mathcal{C}^V$, let v_C be as defined above. And for each $C \in \mathcal{C}^F$, let f_C be as defined above too.

For each $C \in \mathcal{C}^V$, $v_C \in V_\phi$ and $v_C \sqsubseteq v_\phi$. Therefore, by theorem C.1, $v_\phi \in V_\phi$. An entirely analogous argument shows that $f_\phi \in F_\phi$. $\square$

E Entailment and Containment

In this section, I present a few results concerning entailment, containment, and parthood. The following two theorems show that quantifiers, and their instances, stand in the intuitively appropriate entailment relations to one another.

Theorem E.1. *Let χ be a variable, let κ be a constant, and let $\phi(\chi)$ be a formula in which only χ appears free. Then for all states s, if $s \Vdash \forall\chi\phi(\chi)$ then $s \Vdash \phi[\kappa/\chi]$, and if $s \dashv\Vert \phi[\kappa/\chi]$ then $s \dashv\Vert \forall\chi\phi(\chi)$.*

Theorem E.2. *Let χ be a variable, let κ be a constant, and let $\phi(\chi)$ be a formula in which only χ appears free. Then for all states s, if $s \Vdash \phi[\kappa/\chi]$ then $s \Vdash \exists\chi\phi(\chi)$, and if $s \dashv\Vert \exists\chi\phi(\chi)$ then $s \dashv\Vert \phi[\kappa/\chi]$.*

Since the proofs of these theorems are fairly straightforward but also fairly technical,[40] I omit them.

Now for some important facts about parthood and the containment relation. The following theorem says that parthood is sufficient for containment.

Theorem E.3. *Let $\mathcal{P} = \langle \mathcal{P}_V, \mathcal{P}_F \rangle$ and $\mathcal{Q} = \langle \mathcal{Q}_V, \mathcal{Q}_F \rangle$. Suppose that $\mathcal{P}$ is part of $\mathcal{Q}$. Then $\mathcal{P}$ is contained in $\mathcal{Q}$.*

Proof. By theorem C.2, there exists a set of cones C' such that for all $C_\mathcal{Q} \in C'$, $C_\mathcal{Q}$ is maximal in $\mathcal{Q}_V$ and $\mathcal{Q}_V = \bigcup C'$. For each $C_\mathcal{Q} \in C'$, let $c_\mathcal{Q}$ be the point of $C_\mathcal{Q}$. By Partial Content

[40] They rely on the second and third constraints on the functions $|\cdot|^+_\cdot$ and $|\cdot|^-_\cdot$ mentioned in Appendix B.

and Positive Partial Content, it follows that for each $C_{\mathcal{Q}} \in C'$ with point $c_{\mathcal{Q}}$, there is a cone $C_{\mathcal{P}}$ such that (i) $C_{\mathcal{P}}$ is maximal in $\mathcal{P}_V$, and (ii) $C_{\mathcal{P}}$ has a point $c_{\mathcal{P}}$ such that $c_{\mathcal{P}} \sqsubseteq c_{\mathcal{Q}}$. By Proposition, $\mathcal{P}_V$ is upward closed. Therefore, for each $C_{\mathcal{Q}} \in C'$, there is a cone $C_{\mathcal{P}}$ such that (i) $C_{\mathcal{P}}$ is maximal in $\mathcal{P}_V$, and (ii) $C_{\mathcal{Q}} \subseteq C_{\mathcal{P}}$. Therefore, $\mathcal{Q}_V = \bigcup C' \subseteq \mathcal{P}_V$. An entirely analogous argument—using Negative Partial Content—shows that $\mathcal{Q}_F \subseteq \mathcal{P}_F$. Therefore, by Containment, $\mathcal{P}$ is contained in $\mathcal{Q}$. □

The following theorem says that containment is not sufficient for parthood.

Theorem E.4. *There are propositions $\mathcal{P} = \langle \mathcal{P}_V, \mathcal{P}_F \rangle$ and $\mathcal{Q} = \langle \mathcal{Q}_V, \mathcal{Q}_F \rangle$ such that $\mathcal{P}$ is contained in $\mathcal{Q}$ but $\mathcal{P}$ is not part of $\mathcal{Q}$.*

Proof. Let p_1 and p_2 be states such that neither is part of the other. Let p_3 be a state. Let $C_{p_1} = \{s \in S \mid p_1 \sqsubseteq s\}$, $C_{p_2} = \{s \in S \mid p_2 \sqsubseteq s\}$, and $C_{p_3} = \{s \in S \mid p_3 \sqsubseteq s\}$. Let $\mathcal{P}_V = C_{p_1} \cup C_{p_2}$, let $\mathcal{P}_F = C_{p_3}$, let $\mathcal{Q}_V = C_{p_1}$, and let $\mathcal{Q}_F = C_{p_3}$. Then $\mathcal{Q}_V \subseteq \mathcal{P}_V$ and $\mathcal{Q}_F \subseteq \mathcal{P}_F$; so by Containment, $\mathcal{P}$ is contained in $\mathcal{Q}$. Nevertheless, there is a cone $C_{\mathcal{P}}$ which is maximal in $\mathcal{P}_V$ but which has a point $c_{\mathcal{P}}$ such that for all cones $C_{\mathcal{Q}}$ which are maximal in $\mathcal{Q}_V$ and which have a point $c_{\mathcal{Q}}$, $c_{\mathcal{P}} \not\sqsubseteq c_{\mathcal{Q}}$: just let $C_{\mathcal{P}}$ be the cone C_{p_2}. So by Positive Partial Content and Partial Content, $\mathcal{P}$ is not part of $\mathcal{Q}$. □

F Counterfactuals

The comparative similarity relation is expressed by a three-place predicate 'C' over a set of states S. Intuitively, for all $s, t, u \in S$, '$Cstu$' says that s is more similar to t than to u. The relation C varies from context to context: in some contexts, '$Cstu$' might be true; in other contexts, '$Csut$' might be true instead.

G Logical Subtraction

Let $(S, \sqsubseteq)$ be a complete, well-founded lattice. Then here are two more conditions which $(S, \sqsubseteq)$ may satisfy.

1. Distributivity: for all $A \subseteq S$, and for all $s \in S$, $\bigsqcup(A \sqcap s) \sqsubseteq \left(\bigsqcup A\right) \sqcap s$.
2. Existence of Complement: for all $s \in S$, there exists $s^\perp \in S$ such that $s \sqcap s^\perp = 1$ and $s \sqcup s^\perp = 0$.[41]

Now for an extremely important lemma.

Lemma G.1. *Let $(S, \sqsubseteq)$ be a complete, complemented, distributive lattice. Then for all $x, y \in S$ such that $x \sqsubseteq y$, there exists $z \in S$ such that $y = z \sqcap x$.*

Proof. Since $(S, \sqsubseteq)$ is complemented, there exists a state $x^\perp$ such that $x \sqcap x^\perp = 1$ and $x \sqcup x^\perp = 0$. Let $z = y \sqcup x^\perp$. Then $z \sqcap x = (y \sqcup x^\perp) \sqcap x = (y \sqcap x) \sqcup (x^\perp \sqcap x) = (y \sqcap x) \sqcup 1 = (y \sqcap x) = y$, where the first equality follows from the definition of z, the second equality follows from the fact that $(S, \sqsubseteq)$ is distributive, the third equality follows from the definition of $x^\perp$, the fourth equality follows from the definition of 1, and the fifth equality follows from the fact that $(S, \sqsubseteq)$ is a lattice. □

Now for a key result.

Theorem G.1 (theorem 3 in the main text)**.** *Let $(S, \sqsubseteq)$ be a complete, complemented, distributive, well-founded lattice. Let $x, y \in S$ be such that $x \sqsubseteq y$. Let $Z_{x,y} = \{z \in S \mid y = z \sqcap x\}$. Then $Z_{x,y}$ is non-empty and $y = \left(\bigsqcup Z_{x,y}\right) \sqcap x$.*

Proof. Lemma G.1 implies that $Z_{x,y}$ is non-empty. As for the rest of this theorem, note that for each $z \in Z_{x,y}$, $y = z \sqcap x$. Therefore, $Z_{x,y} \sqcap x = \{y\}$. And so $\left(\bigsqcup Z_{x,y}\right) \sqcap x = \bigsqcup(Z_{x,y} \sqcap x) = y$, where the first equality follows from the fact that $(S, \sqsubseteq)$ is distributive, and the second equality follows from the fact that $Z_{x,y} \sqcap x = \{y\}$. □

[41] Note that $0 = \bigsqcup S$ and $1 = \bigsqcap S$.

In what follows, let '$-'$' be the two-place function defined like this: for all states $x, y \in S$ such that $x \sqsubseteq y$, $y -' x = \bigsqcup Z_{x,y}$. Theorem G.1 shows that $-'$ is a well-defined function whose domain is the set of all pairs (x, y) of states in S such that y contains x as a part.

The analysis of logical subtraction is based on several preliminary definitions. To start, let $\mathcal{P} = \langle \mathcal{P}_V, \mathcal{P}_F \rangle$ and $\mathcal{Q} = \langle \mathcal{Q}_V, \mathcal{Q}_F \rangle$ be propositions, and suppose that $\mathcal{P}$ is part of $\mathcal{Q}$. Let $c_1^{\mathcal{P}_V}$, $c_2^{\mathcal{P}_V}$, ..., be the points of the maximal cones in $\mathcal{P}_V$. Let $c_1^{\mathcal{Q}_V}$, $c_2^{\mathcal{Q}_V}$, ..., be the points of the maximal cones in $\mathcal{Q}_V$. Let $c_1^{\mathcal{P}_F}$, $c_2^{\mathcal{P}_F}$, ..., be the points of the maximal cones in $\mathcal{P}_F$. And let $c_1^{\mathcal{Q}_F}$, $c_2^{\mathcal{Q}_F}$, ..., be the points of the maximal cones in $\mathcal{Q}_F$. For each $c_i^{\mathcal{P}_V}$, let $c_{i_1}^{\mathcal{Q}_V}$, $c_{i_2}^{\mathcal{Q}_V}$, ..., be the collection of points of maximal cones in $\mathcal{Q}_V$ such that each $c_{i_j}^{\mathcal{Q}_V}$ contains $c_i^{\mathcal{P}_V}$ as a part. And for each $c_i^{\mathcal{P}_F}$, let $c_{i_1}^{\mathcal{Q}_F}$, $c_{i_2}^{\mathcal{Q}_F}$, ..., be the collection of points of maximal cones in $\mathcal{Q}_F$ such that each $c_{i_j}^{\mathcal{Q}_F}$ contains $c_i^{\mathcal{P}_F}$ as a part. For each pair of states $c_i^{\mathcal{P}_V}$ and $c_{i_j}^{\mathcal{Q}_V}$, let $v_i^{i_j} = c_{i_j}^{\mathcal{Q}_V} -' c_i^{\mathcal{P}_V}$. And for each pair of states $c_i^{\mathcal{P}_F}$ and $c_{i_j}^{\mathcal{Q}_F}$, let $f_i^{i_j} = c_{i_j}^{\mathcal{Q}_F} -' c_i^{\mathcal{P}_F}$. Finally, define '$-^*$' as follows: let $\mathcal{Q}_V -^* \mathcal{P}_V = \{s \in S \mid \text{for some } i \text{ and } j,\ v_i^{i_j} \sqsubseteq s\}$, and let $\mathcal{Q}_F -^* \mathcal{P}_F = \{s \in S \mid \text{for some } i \text{ and } j,\ f_i^{i_j} \sqsubseteq s\}$. Then the operation '$-$' of logical subtraction may be defined as in the main text.

Acknowledgements

[to be added]

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