Handout (Week 2)

The Formal Language

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Goals of this handout: introduce the formal language L of propositional logic, and explain how L can be used to represent natural language sentences.

Reading guide for completing the homework

- To solve homework problems 1-15, you will probably have to read all of sections 1.1–1.8, 3.1, and 3.2.

Summary of handout

- Sections 1.1–1.7: I present the vocabulary of L. Along the way, I explain what that vocabulary means, by using it to translate various English sentences and expressions.
- Section 1.8: I summarize the vocabulary of L, so that you can locate it quickly when doing the homework problems.
- Section 2: I provide some more examples of English-language expressions which can be translated using the vocabulary of L.
- Section 3.1: I define the sentences of L.
- Section 3.2: I give several examples of sentences of L, using the definition of a sentence.
- Section 3.3: I introduce some more standard notation and terminology.

1 The Formal Language

We will now study the language of propositional logic; I call it 'L'. This language has just a few different kinds of symbols. But those symbols can be used to formulate an extremely large range of different kinds of sentences.

To give you a feel for L—and to explain why it is useful—I will connect it to a language with which you are more familiar. In particular, I will show how various symbols in L can be used to represent English sentences. So to start, I will introduce L slowly, one symbol at a time. After that, I will give a quick and complete characterization of all the symbols of L. Finally, I will say what it takes for a sequence of symbols in L to be a grammatical, well-formulated sentence.

An important aside: as will become clear, the very same expressions in L—the very same strings of symbols—can be used to represent many, many different sentences of English. This is a pretty common feature of languages in general. For instance, many different English sentences can be used to represent—or translate—a German sentence like "Es regnet": this sentence can be represented by "It is raining" or "It rains", for instance. So when working with L, keep in mind that one and the same expression can represent, or be represented by, multiple English sentences.

1.1 Sentence Letters

L includes a bunch of letters: for instance, it contains 'p' and 'q'. Call these symbols 'sentence letters'. Here are some examples of how sentence letters can be used to represent English sentences.

> Example
> The symbol 'p' can be used to represent the English sentence "It is raining".

In other words, we can use L to translate "It is raining" as 'p'.

Example

The symbol 'p' can be used to represent the English sentence "The final is on Thursday". In other words, we can use L to translate "The final is on Thursday" as 'p'.

Example

The symbol 'p' can be used to represent the English sentence "The final is on Thursday", and the symbol 'q' can be used to represent the English sentence "The final is on Friday". In other words, we can use L to translate "The final is on Thursday" as 'p' and "The final is on Friday" as 'q'.

An important point about the last example: since the two English sentences—namely, "The final is on Thursday" and "The final is on Friday"—mean different things, the sentence letters used to represent them should be different as well. It would be bad to translate "The final is on Tuesday" as 'p', and then to translate "The final is on Friday" as 'p' too. For then the very same symbol—namely, 'p'—would have multiple meanings. All else equal, it is better for expressions of a language to mean just one thing: that cuts down on ambiguity and confusion. So here is a rule to remember: in any given problem or example, if two English sentences have different meanings, then use different sentence letters to represent those sentences.

L contains infinitely many sentence letters: 'p', 'q', 'r', 's', 't', and many, many more. In all of the examples to come, I will never use more than five sentence letters. But it is worth pointing out that L contains many more sentence letters than just five. To symbolize them all, just appeal to subscripts: for instance, the infinitude of sentence letters could be expressed as 'p_1', 'p_2', 'p_3', and so on.

1.2 Parentheses

L includes the parentheses '(' and ')'. Parentheses are not used to translate English words. Instead, parentheses are used to organize the other symbols of L, in order to be completely precise about exactly what the sentence in question says. In the later subsections, it will become clear how parentheses are used, and why their organizational role is so important.

1.3 Negation

L includes a negation symbol: '$\neg$'. This symbol represents the English word 'not'. Think of negation as the formal translation of this English-language construction: "It is not the case that . . . ".

Here is an example of how '$\neg$' can be used to represent English sentences.

> Example
> The expression '$\neg p$' can be used to represent the English sentence "It is not raining". In other words, we can use L to translate "It is not raining" as '$\neg p$'.

Note that in this example, the translation of "It is not raining" can be broken into two steps. First, we used 'p' to represent "It is raining". Then we negated 'p', in order to represent the negation of that English sentence. As a result, we used '$\neg p$' to represent "It is not raining".

Now for another example.

> Example
> The expression '$\neg p$' can be used to represent the English sentence "It is not

> the case that the final is on Thursday". In other words, we can use L to translate "It is not the case that the final is on Thursday" as '$\neg p$'.

Again, the translation can be broken into two steps. First, we used 'p' to represent "The final is on Thursday". Then we negated 'p', in order to represent the negation of that English sentence. As a result, we used '$\neg p$' to represent "It is not the case that the final is on Thursday".

Here is one final example.

> Example
> The expression '$\neg\neg p$' can be used to represent the English sentence "It is not the case that it is not the case that the final is on Friday". In other words, we can use L to translate "It is not the case that it is not the case that the final is on Friday" as '$\neg\neg p$'.

This translation can be broken into *three* steps. First, we used 'p' to represent "The final is on Friday". Then we negated 'p' once, in order to represent the negation of that English sentence. As a result, we implicitly took '$\neg p$' to represent "It is not the case that the final is on Friday". Finally, we negated '$\neg p$', in order to represent the negation of the English sentence "It is not the case that the final is on Friday". As a result, we used '$\neg\neg p$' to represent "It is not the case that it is not the case that the final is on Friday".

Think of the English sentence, in this example, as a more complicated way of saying "The final is not *not* on Friday". Given that, we can think of "The final is not *not* on Friday" as represented by '$\neg\neg p$'.

1.4 Conjunction

L includes a conjunction symbol: '$\wedge$'. This symbol represents the English word 'and'. Think of conjunction as the formal translation of this linguistic construction: "... and ...".

Here is an example of how '$\wedge$' can be used to represent English sentences.

> Example
> The expression '$p \wedge q$' can be used to represent the English sentence "The final is on Thursday and the final is on Friday".

In this translation, the 'p' stands for "The final is on Thursday" and the 'q' stands for 'The final is on Friday". The '$\wedge$' is used to translate the 'and'.

This example illustrates why different English sentences should not be represented with the same sentence letter. It we did that here, the result would be, say, '$p \wedge p$', which clearly is not what the English sentence means. The sentence '$p \wedge p$' means something more like "The final is on Thursday and the final is on Thursday".

Here are two more examples.

> Example
> The expression '$p \wedge q$' can be used to represent the English sentence "It is raining and there are clouds in the sky".

> Example
> The expression '$p \wedge \neg q$' can be used to represent the English sentence "The final is on Thursday and not Friday".

On the above translation, 'p' represents "The final is on Thursday" and 'q' represents "The final is on Friday". And so '$\neg q$' represents "The final is not on Friday", which means the same thing as the "not Friday" clause in the example.

An alternative—but worse—translation would be this: '$p \wedge q$', where this time, 'q' represents "The final is not on Friday". This is a suboptimal translation of the original English sentence, because it misses out on some of that sentence's *logical structure*. In particular, it misses out on the structure captured by the '$\neg$'. Whenever it is possible to translate an English sentence in a way that uses '$\neg$', '$\wedge$', or some other logical symbol of L, do it. For the resulting translation gives you a more precise representation of the logical features of the original English sentence.

Here is a final example.

> Example
> The expression '$\neg(p \wedge q)$' can be used to represent the English sentence "It is not the case that the final is on Thursday and Friday".

Note that this example includes some parentheses. Those parentheses are extremely important: they play a crucial role in correctly capturing the meaning of the English sentence "It is not the case that the final is on Thursday and Friday". To see why, just observe what happens when we remove them. The resulting expression is '$\neg p \wedge q$'. But '$\neg p \wedge q$' does not mean the same thing as '$\neg(p \wedge q)$'. In particular, '$\neg p \wedge q$' does not represent the English sentence "It is not the case that the final is on Thursday and Friday". Rather, '$\neg p \wedge q$' represents the English sentence "It is not the case that the final is on Thursday, and the final is on Friday".

It is worth driving this subtle point home. So just to be clear, here are two sentences in L which (i) are distinguished only by how they use parentheses, and (ii) represent different sorts of English sentences.

1. '$\neg(p \wedge q)$': this represents sentences like the following.

- "It is not the case that the final is on Thursday and Friday".
- "It is not the case that the final is on both Thursday and Friday".
- "The final is not both on Thursday and also on Friday".

2. '$\neg p \wedge q$': this represents sentences like the following.
    - "It is not the case that the final is on Thursday, and the final is on Friday".
    - "It is not the case that the final is on Thursday, but note that the final is on Friday".
    - "It is not the case that the final is on Thursday, and also, by the way, the final is on Friday".

In short, '$\neg(p \wedge q)$' says that the final is not on *both* days, while '$\neg p \wedge q$' says that the final is *not* on the first day (Thursday) but *is* on the second day (Friday).

1.5 Disjunction

L includes a disjunction symbol: '$\vee$'. This symbol represents the English word 'or'. Think of disjunction as the formal translation of this linguistic construction: "... or ...".

Here are some examples of how '$\vee$' can be used to represent English sentences.

Example
The expression '$p \vee q$' can be used to represent the English sentence "It is raining or there are clouds in the sky".

Example
The expression '$p \vee \neg q$' can be used to represent the English sentence "The final is on Thursday or the final is not on Friday".

Example

The expression '$p \vee (q \wedge r)$' can be used to represent the English sentence "Either the final is on Thursday, or the final is on Friday and violets are blue".

In the example above, 'p' represents "The final is on Thursday", 'q' represents "The final is on Friday", and 'r' represents "Violets are blue".

Here is a final example.

Example
The expression '$p \vee (\neg q \wedge r)$' can be used to represent "The final is on Thursday, or it is not raining and there are clouds in the sky".

In the example above, 'p' represents "The final is on Thursday". The sentence letter 'q' represents "It is raining"; so '$\neg q$' represents "It is not raining". Finally, 'r' represents "There are clouds in the sky"; so '$\neg q \wedge r$' represents "It is not raining and there are clouds in the sky".

1.6 Conditional

L includes a conditional symbol: '$\rightarrow$'. This symbol represents the following English-language construction: "If ... then ...".

Here are some examples of how '$\rightarrow$' can be used to represent English sentences.

Example
The expression '$p \rightarrow q$' can be used to represent "If it is raining then there are clouds in the sky".

Example
The expression '$\neg q \rightarrow \neg p$' can be used to represent "If there are no clouds

in the sky, then it is not raining".

In both of the examples above, 'p' represents "It is raining" and 'q' represents "There are clouds in the sky".

Here is a final example.

Example
The expression '$(p \wedge \neg q) \rightarrow r$' can be used to represent "If it is raining and I do not have an umbrella, then I will get wet".

In this example, 'p' represents "It is raining". The sentence letter 'q' represents "I have an umbrella"; so '$\neg q$' represents "I do not have an umbrella". And so '$p \wedge \neg q$' represents "It is raining and I do not have an umbrella". Finally, 'r' represents "I will get wet".

1.7 Biconditional

L includes a biconditional symbol: '$\leftrightarrow$'. This symbol represents the following English-language construction: "... if and only if ...".

Here is an example of how '$\leftrightarrow$' can be used to represent English sentences.

Example
The expression '$p \leftrightarrow q$' can be used to represent "It is raining if and only if there are clouds in the sky".

In this example, 'p' represents "It is raining" and 'q' represents "There are clouds in the sky".

Here is a final example.

> Example
> The expression '$p \leftrightarrow \neg q$' can be used to represent "Susie will go to the party if and only if Avon will not go".

In this example, 'p' represents "Susie will go to the party". The sentence letter 'q' represents "Avon will go to the party". As a result, '$\neg q$' represents "Avon will not go to the party", which means the same thing as the sentence "Avon will not go" in the example.

1.8 Summary

Here is a summary of all the symbols which the formal language L contains.

1. Sentence letters: 'p', 'q', 'r', 's', 't', and so on.
2. Parentheses: '(' and ')'.
3. Negation: '$\neg$'.
4. Conjunction: '$\wedge$'.
5. Disjunction: '$\vee$'.
6. Conditional: '$\rightarrow$'.
7. Biconiditional: '$\leftrightarrow$'.

The symbols '$\neg$', '$\wedge$', '$\vee$', '$\rightarrow$', and '$\leftrightarrow$' are called 'connectives'. This is because they *connect* to sentences of L to form other sentences of L. Sometimes they are also called 'logical connectives', 'logical symbols', or 'logical constants'. The reasons for this are somewhat involved, but basically, it is because they always mean the same thing. Whereas 'p' can mean different things, depending on what it is used to translate, the meanings of '$\neg$', '$\wedge$', '$\vee$', '$\rightarrow$', and '$\leftrightarrow$', are constant.

2 Translation Tips

Throughout Section 1, I explained the meaning of each symbol '$\neg$', '$\wedge$', '$\vee$', '$\rightarrow$', and '$\leftrightarrow$', using just one English word or phrase. For instance, I wrote that '$\neg$' represents the English word 'not', and I wrote that '$\wedge$' represents the English word 'and'.

But actually, symbols like '$\neg$' and '$\wedge$' can be used to represent many more English words. Here is an *incomplete* list of the sorts of English words, phrases, expressions, and so on, which symbols in L can help represent.

1. '$\neg$': 'not', 'It is not the case that...', 'neither... nor...', and any other word or expression which suggests that something is being denied.
2. '$\wedge$': 'and', 'Both... and...', 'but', 'Though..., ...', and any other word or expression which suggests that some things have been conjoined.
3. '$\vee$': 'or', 'Either... or...', 'neither... nor...', and any other word or expression which suggests that some things have been disjoined.
4. '$\rightarrow$': 'If ... then ...', 'When ..., ...', 'Since ..., ...', and any other word or expression which suggests that one thing follows from another thing.
5. '$\leftrightarrow$': '... if and only if ...', '... just in case ...', '... exactly when ...', and any other word or expression which suggests that (i) one thing follows from another thing, and (ii) that other thing follows from the first thing.

There is no fully precise, fully rigorous way to say exactly which English expressions can be represented using the symbols of L. That is a common feature of all languages: there is no fully precise, fully rigorous way to say exactly which German expressions can be represented using the English word 'not', for instance. But as you work with L, you will get a better and better sense for how its symbols relate to various English language constructions. So be sure to carefully study the above examples of sentences in L which represent English sentences.

3 Sentences of L

In this section, I give a fully rigorous definition of what counts as a sentence of L. Then I give a series of examples of that definition at work. Finally, I briefly mention some notation and terminology which will be used throughout the rest of this course.

3.1 The Recursive Definition of a Sentence of L

The definition of a sentence of L has roughly three parts. In the first and simplest part, the definition says what the basic, simplest sentences of L are. In the second and most complicated part, the definition says how to use simpler sentences—along with symbols like '$\neg$', '$\wedge$', and so on—to construct more complicated sentences. In the third part, the definition says that there are no other ways to get sentences in L: the only sentences are the ones that can be built using the first part and the second part.

Here is the definition; call it the 'Recursive Definition' of a sentence of L.

Definition: Sentence of L
The 'sentences' of L are defined as follows.

1. First part
   - Every sentence letter is a sentence.
2. Second part
   - If 'ϕ' is a sentence, then '$\neg\phi$' is a sentence.
   - If 'ϕ' and 'ψ' are sentences, then '$(\phi \wedge \psi)$' is a sentence.
   - If 'ϕ' and 'ψ' are sentences, then '$(\phi \vee \psi)$' is a sentence.
   - If 'ϕ' and 'ψ' are sentences, then '$(\phi \rightarrow \psi)$' is a sentence.
   - If 'ϕ' and 'ψ' are sentences, then '$(\phi \leftrightarrow \psi)$' is a sentence.
3. Third part

- Nothing else is a sentence.

In general, the sentences of a language—whether that language is English, L, or something else—are the grammatical, well-formulated, coherent expressions of that language. In any language, many expressions are not grammatical: for instance, "Red that coffee is and" is an English expression, since it contains only English words; yet it is not grammatical, since those words do not conform to the English rules for sentence formation. Just like English, L has rules for forming sentences. And just like English, the sentences of L are the expressions of L which conform to those rules.

3.2 Examples

In this section, I present three examples of the Recursive Definition of a sentence of L.

For a particularly simple example, note that 'p' is a sentence of L. This follows from (i) the first part of the Recursive Definition, and (ii) the fact that 'p' is a sentence letter.

For a more complicated example, consider '$(p \wedge q)$'. This too is a sentence of L. To see why, note that since both 'p' and 'q' are sentence letters, the first part of the Recursive Definition implies that both 'p' and 'q' are sentences. Now consider the *second* bullet point in the *second* part of the Recursive Definition: it says that if 'ϕ' and 'ψ' are sentences, then '$(\phi \wedge \psi)$' is a sentence. Substituting 'p' for 'ϕ' and 'q' for 'ψ', we get the following: if 'p' and 'q' are sentences, then '$(p \wedge q)$' is a sentence. As I just showed, 'p' and 'q' are indeed sentences of L. So it follows that '$(p \wedge q)$' is a sentence.

For an even more complicated example, consider '$((p \wedge q) \rightarrow r)$'. This is a sentence of L too. As I just showed, '$(p \wedge q)$' is a sentence. Since 'r' is a sentence letter, the first part of the Recursive Definition implies that 'r' is a sentence of L. Now consider the *fourth* bullet point in the *second* part of the Recursive Definition: it says that if 'ϕ' and 'ψ' are sentences, then '$(\phi \rightarrow \psi)$' is a sentence. Substituting '$(p \wedge q)$' for 'ϕ' and 'r' for 'ψ', we get the following: if '$(p \wedge q)$' and 'r' are sentences, then '$((p \wedge q) \rightarrow r)$' is a sentence. As I just showed, '$(p \wedge q)$' and 'r' are indeed sentences. Therefore, '$((p \wedge q) \rightarrow r)$' is a sentence.

Plenty of expressions of L—plenty of strings of symbols in L, that is—are *not* sentences. Here is one: 'pq'. This is not a sentence because (i) it is not a sentence letter, and (ii) it cannot be built from sentences using the five bullet points in the second part of the Recursive Definition. Here is another: '$p \rightarrow$'. Again, this is not a sentence because (i) it is not a sentence letter, and (ii) it cannot be built from sentences using the five bullet points in the second part of the Recursive Definition. Here is another: '$p \rightarrow (\neg \vee r)$'. This is not a sentence for the same reason: it is not a sentence letter, and it cannot be built from sentences using those five bullet points.

3.3 Notation and Terminology

Before closing, it is worth introducing some standard notation and terminology. Here is one notational shorthand: following standard practice, I will often omit the outermost parentheses of sentences of L. For instance, instead of writing '$(p \wedge q)$', I will write '$p \wedge q$'. And instead of writing '$((p \wedge q) \rightarrow r)$', I will write '$(p \wedge q) \rightarrow r$'. This is standard practice because it increases readability: after you work with L for a while, it becomes easier to parse expressions like '$(((p \wedge q) \rightarrow r) \vee (s \leftrightarrow \neg(q \rightarrow p)))$' if the outermost parentheses are dropped, and thus, if those expressions are rewritten as '$((p \wedge q) \rightarrow r) \vee (s \leftrightarrow \neg(q \rightarrow p))$'. So in this course, we will count expressions like '$p \wedge q$' as sentences, even though—contrary to the second bullet point in the Recursive Definition—they do not contain outermost parentheses.

Now for some terminology. In a sentence of the form '$\phi \wedge \psi$', 'ϕ' and 'ψ' are called the 'conjuncts' of '$\phi \wedge \psi$'. For they are terms in that conjunction. Likewise, in an English sentence such as "It is raining and there are clouds in the sky", the sentences on either side of 'and' are called 'conjuncts' of that English sentence too. "It is raining" is a conjunct of that sentence, for instance, as is "There are clouds in the sky".

Similarly, in a sentence of the form '$\phi \vee \psi$', 'ϕ' and 'ψ' are called 'disjuncts' of '$\phi \vee \psi$'. For they are terms in that disjunction. Likewise, in an English sentence such as "It is raining or there are clouds in the sky", the sentences on either side of 'or' are called 'disjuncts' of that English sentence too. "It is raining" is a disjunct of that sentence, for instance, as is "There are clouds in the sky".

Finally, in a sentence of the form '$\phi \rightarrow \psi$', 'ϕ' is the 'antecedent' of '$\phi \rightarrow \psi$', and 'ψ' is the 'consequent' of '$\phi \rightarrow \psi$'. Likewise, in an English sentence such as "If it is raining, then there are clouds in the sky", "It is raining" is the antecedent of that sentence and "There are clouds in the sky" is the consequent of that sentence.